Maxwell's equation

Divergence ($\vec{\nabla}.\_$) and curl $\vec{\nabla}\times\_$) relations of electric (E) and magnetic field (B) Electromagnetic Wave are called Maxwell's equation.

Differential form of Maxwell's equation

1. **First equation-** It is **differential form of Gauss Law in electrostatic**.
   Divergence of electric displacement field (electric flux density) -
   $(\vec{\nabla}.\vec{D} = \rho)$ (A)
   Derivation
   According to Gauss's law for closed surface
   $\iint \vec{E}.\vec{ds} = \frac{q}{\epsilon_0}$ (1)
   If density of uniform charge distribution at any point is ρ, then
   $q = \iiint \rho dv$ (2)
   From eq. (1) and (2), $\iint \vec{E}.\vec{ds} = \frac{1}{\epsilon_0}\iiint \rho dv$ (3)
   According to gauss divergence theorem, $\iint \vec{E}.\vec{ds} = \iiint(\vec{\nabla}.\vec{E})dv$ (4)
   So from (3) and (4), $\iiint(\vec{\nabla}.\vec{E})dv = \frac{1}{\epsilon_0}\iiint \rho dv$ (5)
   $\boxed{Div\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{or} \quad Div\vec{D} = \rho}$ (6)
2. **Second equation-** It is differential form of Gauss Law in magneto static.($\vec{\nabla}.\vec{B} = 0$)
   The number of lines of forces entering any surface is exactly equal to that leaving the surface, i.e. "The net magnetic flux through a closed surface is always zero". ($\iint \vec{B}.\vec{ds} = 0$) (B)
   According to gauss divergence theorem, $\iint \vec{B}.\vec{ds} = \iiint(\vec{\nabla}.\vec{B})dv$ (1)
   This equation valid for any arbitrary volume, so the integration must be zero, so from equation (B) and (1), $\boxed{Div\vec{B} = 0 \ or\ \vec{\nabla}.\vec{B} = 0}$ (2)
3. **Third equation-** It is Faraday's law of electromagnetic induction.(curl $\vec{E} = -\frac{\partial \vec{B}}{\partial t}$) (C)
   according to Faraday's law, the induced. e.m.f. ($e = \oint \vec{E}.\vec{dl}$) produced in a closed ciecuit is equal to the negative of the rate change of magnetic flux linked with that circuit. i.e.
   $\vec{e} = -\frac{d\emptyset}{dt} = -\frac{d}{dt}\iint \vec{B}.\vec{ds}$ (1)
   And $(e = \oint \vec{E}.\vec{dl})$ (2) from equation (1) and (2)
   $\oint \vec{E}.\vec{dl} = -\frac{d}{dt}\iint \vec{B}.\vec{ds}$ **Or** $\oint \vec{E}.\vec{dl} = -\iint \frac{\partial \vec{B}}{\partial t}.\vec{ds}$ (3)
   According to Stoke's theorem, $\oint \vec{E}.\vec{dl} = \iint(curl\,\vec{E}).\vec{ds}$ (4)
   so, from equation (3) and (4), $\iint(curl\,\vec{E}).\vec{ds} = -\iint \frac{\partial\vec{B}}{\partial t}.\vec{ds}$ (5)
   the equation valid for any arbitrary surface, hence both the vectors in the integral must be equal at each point. So (6)
   $$\boxed{curl\,\vec{E} = -\frac{\partial\vec{B}}{\partial t} \ or\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Maxwell's equation

4. **Fourth equation-** It is modified Ampere's law, which is modified by Maxwell.

$$curl\,\vec{B} = \mu_0\left[\vec{j} + \frac{\partial\vec{D}}{\partial t}\right] \qquad \text{(D)}$$

According to Ampere's law,

$$\oint \vec{B}.\vec{dl} = \mu_0\, I = \mu_0 \iint \vec{J}.\vec{ds} \qquad (1)$$

According to Stoke's theorem, $\oint \vec{B}.\vec{dl} = \iint (curl\,\vec{B}).\vec{ds}$ (2)

Hence, $\iint (curl\,\vec{B}).\vec{ds} = \mu_0 \iint \vec{J}.\vec{ds}$ (3)

so that, $curl\,\vec{B} = \mu_0\vec{J}$ (4)

This equation valid for steady current for varying electric fields, $Div\vec{J} + \frac{\partial\rho}{\partial t} = 0$ (from eq. of continuity), taking div of equation (4) we get-

$div\ curl\,\vec{B} = \mu_0\ div\ \vec{J}$, But $\vec{\nabla}.(\vec{\nabla}\times\vec{B}) = 0$, so $div\ \vec{J} = 0$

Equation (1) obtain from Ampere's law , is not accordance with the equation of continuity, Hence it need corrections, According to Maxwell eq.(2) is in complete for the definition of total current density. For this Maxwell suggested that we must add some vector $\vec{J}_d$ to it. Then total current density must be solenoid. i.e.-

$$curl\,\vec{B} = \mu_0\,(\vec{J} + \vec{J}_d) \qquad (5)$$

So, $0 = div(\vec{J} + \vec{J}_d)$ or $-div\,\vec{J} = div\,\vec{J}_d = -\frac{\partial\rho}{\partial t}$

But, $\vec{\nabla}.\vec{D} = \rho$ (from eq.(1)) so, $div\,\vec{J}_d = -\frac{\partial(\vec{\nabla}.\vec{D})}{\partial t}$

Where $\vec{D}$ is electric displacement vector, Hence

$$\frac{\partial(\vec{\nabla}.\vec{D})}{\partial t} = \vec{\nabla}.\frac{\partial\vec{D}}{\partial t} = div\left(\frac{\partial\vec{D}}{\partial t}\right)$$

$div\,\vec{J}_d = div\left(\frac{\partial\vec{D}}{\partial t}\right)$ so that $\vec{J}_d = \frac{\partial\vec{D}}{\partial t}$ (6)

hence equation (5) will be,

$$\boxed{curl\,\vec{B} = \mu_0\left|\vec{j} + \frac{\partial\vec{D}}{\partial t}\right|}$$

Or

$$\boxed{curl\,\vec{B} = \mu_0\left|\vec{j} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right|}$$

Integral form of Maxwell's Equation

First equation- $\iint \vec{E}.\vec{ds} = \frac{q}{\epsilon_0}$ (1)

Second equation- $\iint \vec{B}.\vec{ds} = 0$ (2)

Third equation- $\oint \vec{E}.\vec{dl} = -\iint \frac{\partial\vec{B}}{\partial t}.\vec{ds}$ (3)

Fourth equation- $\oint \vec{B}.\vec{dl} = \mu_0 \iint \left[\vec{j} + \frac{\partial\vec{D}}{\partial t}\right].\vec{ds}$ (4)