CHAPTER 1

INTRODUCTION

Brief History of Graph Theory

One of the most important tools in modern mathematics is the theory of graphs. The development of graph theory was very similar to that of probability theory, where much of the original work was motivated by efforts to understand games of chance. Large portions of graph theory have been motivated by the study of games and recreational mathematics. The origin of graph theory is attributed to a paper by Leonard Euler Published in 1736. He solved the famous Konigsberg seven bridges problem [13]. This later paved the way to the concept of an Eulerian graph. For the next hundred years nothing more was done in this field.

In 1847, Gustuv R. Kirchhoff developed the theory of trees for their applications in electrical networks [10]. Arthur Cayley one of the founding fathers of graph theory, became interested in graph theory for the purpose of counting trees. In 1857, he was trying to count the number of structural isomers of the saturated hydrocarbons C_kH_{2k+2}. He used a connected graph to represent the C_kH_{2k+2} molecules. Corresponding to their chemical valences, a carbon atom was represented by a vertex of degree four and a hydrogen atom by a vertex of degree one (pendant vertex). The total number of vertices in such a graph is

$$n = 3k + 2,$$

and the total number of edges is

$$e = ½\ (\text{sum of degrees}) = ½\ (4k + 2k + 2) = 3k + 1.$$

Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree. Thus the problem of counting structural isomers of a given hydrocarbon becomes the problem of counting trees. In Cayley's classic paper [5], a great deal of work has been done on counting of different types of graphs and results have been applied for solving some practical problems. Later, J. C. Maxwell, James J. Sylvester, George Polya and others used trees to enumerate chemical molecules.

The study of cycles on polyhedra by Thomas P. Kirkman and William R. Hamilton led to the concept of a Hamiltonian graph. At the time of Cayley and Kirchhoff, two milestones were created. One was related to the four color conjecture which states that "four colors are sufficient to color any map on a plane in such a way that no two adjacent regions have the same color" [29]. The other was related to the puzzle put forth by Sir William Rowan Hamilton known as an icosian game [10].

The four color problem was first stated by Francis Gutherie in 1852, and a proof by Alfred Bray Kempe appeared in 1879, which was incorrect. A simpler computational proof was subsequently produced by Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas in 1994 [28]. In 2000 Ashay Dharwarkar offered a totally different proof of the four colour problem. Using complex group theory, Steiner's systems and Hall matchings. The proof was long and turned out be the most refined proof of that time without the use of computer. Later Peter Dorre [30] and F. A. Alhargan [3] gave simpler independent proofs for the four color theorem without using computers. For solving the four colors problem, a map was converted to a graph by treating the regions as vertices and the common boundary as edges. Vertex coloring and edge coloring of graphs forms a major topic in the study of graph theory.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920's. One of the pioneers in this period was D. Konig. He organized the work of other mathematicians and his own and wrote the first book on the subject, which was published in 1936 [10].

The next 30 years were a period of intense activity in graph theory both pure and applied. A great deal of research was done in this area. The main contributions were of Claude Berge, Oystein Ore, Paul Erdos, William Tutte and Frank Harary.

A graph is a very convenient and natural way of representing the relationships between objects, the objects being represented by vertices and the relationship between them by lines.

A graph can also be used as a mathematical model for solving graph-theoretic problems, and then interpreting the solution in terms of the original problem.

Most of the problems in graph theory can be classified under the following categories:

(1) Existence problems
(2) Construction problems
(3) Enumeration problems
(4) Optimization problems

Basic Definitions of Graph Theory [10, 16]

Definition 1.1: Graph

A graph G is an ordered pair of a set of vertices and a set of edges i.e. G = (V, E) where V is the set of vertices and E is the set of two point subset of V. Fig. 1.1 represents a graph with five vertices and six edges.

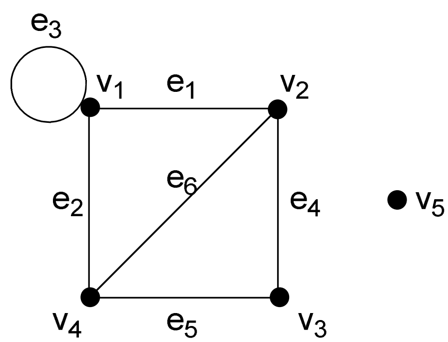

Fig. 1.1: Graph on five vertices

A graph with finite number of vertices and a finite number of edges is called a finite graph. A graph which is not finite is an infinite graph.

Definition 1.2: Directed graph

A directed graph or digraph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices.

In the diagram of a directed graph, each edge $e = (u, v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v. Fig. 1.2 represents a diagraph with three vertices.

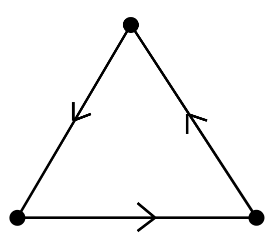

Fig. 1.2: Directed graph

Definition 1.3: Strongly connected diagraph

A diagraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex. A diagraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected. Fig. 1.2 represents the strongly connected diagraph and Fig. 1.3 represents the weakly connected diagraph.

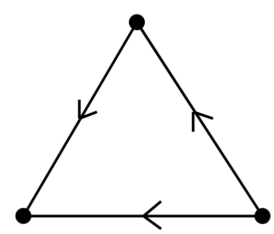

Fig. 1.3: Weakly connected diagraph

Definition 1.4: Independent set

Let G = (V, E) be a graph. A subset A of V is called independent in G if no two vertices of A are end points of same edge in G.

Definition 1.5: Complement of the graph

Let G = (V, E) be a simple undirected graph. The complement of the graph G is the graph with the same set of vertices as G, where two distinct vertices are adjacent if and only if they are independent in G.

Definition 1.6: Order of the graph

The number of vertices in the graph is called the order of the graph. For example order of the graph in Fig. 1.1 is five.

Definition 1.7: Size of the graph

The number of edges in the graph is called the size of the graph. For example size of the graph in Fig. 1.1 is six.

Definition 1.8: Self loop

An edge represented by an unordered pairs in which two elements are not distinct. For example the Fig. 1.1 contains a self loop on the vertex v_1.

Definition 1.9: Parallel edges

Two or more edges in a graph are called parallel edges if they join the same pair of distinct vertices. In Fig. 1.4 e_6 and e_7 are parallel edges.

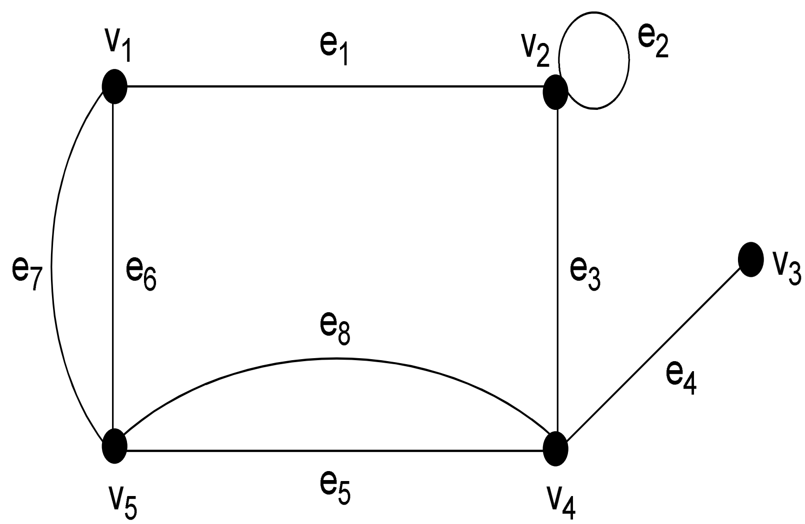

Fig. 1.4: Graph with parallel edges

Definition 1.10: Simple graph

A graph G is called simple if it neither contains parallel edges nor self loops.

Definition 1.11: Degree of vertex

Let v be a vertex in a graph G, then the number of edges incident on v is called the degree of the graph. It is denoted by d(v) or $deg_G(v)$.

Definition 1.12: Maximum degree of the graph

The maximum degree amongst all the vertices v of a graph G is called maximum degree of the graph G. It is denoted by $\Delta = \Delta(G)$.

Definition 1.13: Minimum degree of the graph

The minimum degree amongst all the vertices v of a graph G is called minimum degree of the graph G. It is denoted by $\delta = \delta(G)$.

Definition 1.14: Isolated vertex

A vertex having no incident edge is called an isolated vertex.

Definition 1.15: Pendant or End vertex

A vertex of degree one, is called a pendant vertex or an end vertex. In Fig. 1.4 v_3 is pendant vertex.

Definition 1.16: Adjacent vertices

Two vertices v_i and v_j in a graph G are called adjacent if there is an edge e whose end vertices are v_i and v_j.

Definition 1.17: Trivial graph

A graph which contains only one isolated vertex is called trivial graph.

Fig. 1.5: Trivial graph

Definition 1.18: Null graph

A graph which contains only two or more isolated vertices is called a null graph.

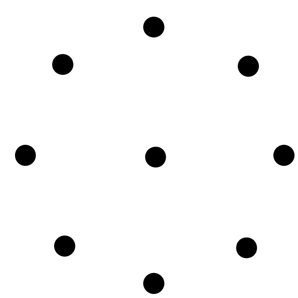

Fig. 1.6: Null graph

Definition 1.19: Connected graph

A graph G is called connected if there exist at least one path between every pair of distinct vertices.

Definition 1.20: Disconnected graph

A graph which contains more than one connected components is called disconnected graph. For example see Fig. 1.1.

Definition 1.21: Complete graph

A simple graph in which there exists an edge between every pair of vertices is called complete graph. The complete graph with n vertices is denoted by K_n. In this graph, degree of every vertex is n-1 and number of edges are ${}^nC_2$.

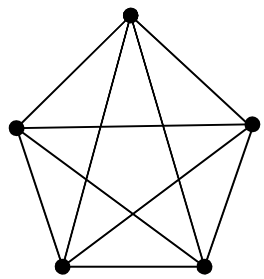

Fig. 1.7: Complete graph with five vertices

Definition 1.22: Regular graph

A graph G is called d- regular if each vertex is of degree d i.e. $d = \Delta = \delta$.

A graph G is not regular is called irregular or non-regular graph. Fig. 1.8 represents the regular graphs with six vertices.

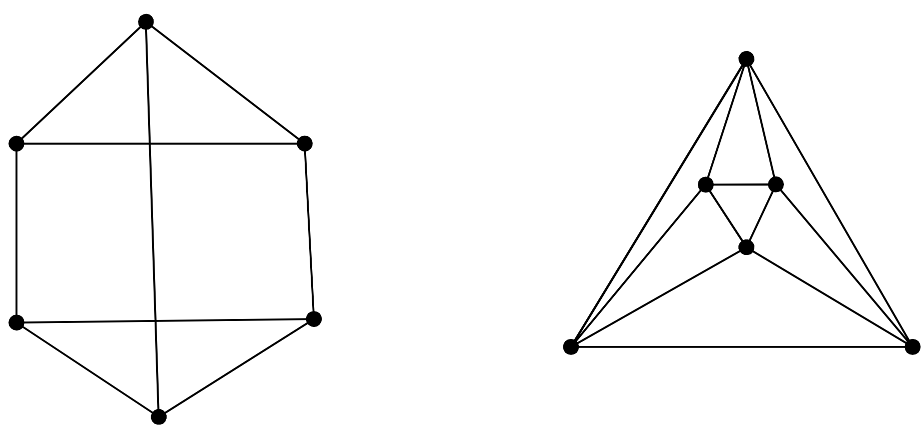

Fig. 1.8: Regular graphs with six vertices

Definition 1.23: Quasi-regular graph

Let G be a connected simple non-regular graph. If each vertex of G has degree either Δ or δ $(\Delta \neq \delta)$, then the graph G is called a quasi-regular.

Definition 1.24: Bidegreed graph

A n-connected simple non-regular graph G is called bidegreed if each vertex of G has degree either δ or $n-1$ $(\delta \neq n-1)$.

Definition 1.25: Bipartite graph

A graph G = (V, E) is called bipartite graph if the vertex set V can be partitioned into two non-empty subsets X and Y (i.e. V = X $\cup$Y) such that each edge $e \in$E has one end vertex in X and other end vertex in Y. In this case, X and Y are called partite sets of the bipartite graph G. It is denoted by G =(X, Y, E).

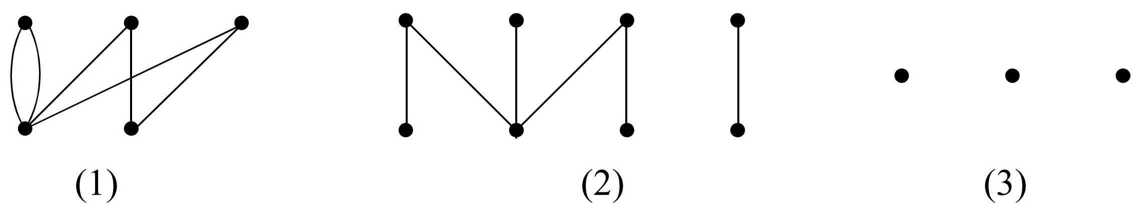

Fig. 1.9: Three different bipartite graphs

Definition 1.26: K-Partite graph

A graph G is called k-partite if the vertex set *V(G)* is the union of k disjoint independent sets.

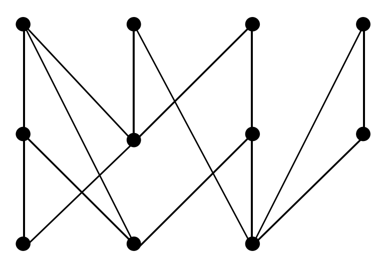

Fig. 1.10: 3-partite graph

Definition 1.27: Complete bipartite graph

Let G = (X, Y, E) be a partite graph with partite sets X and Y such that cardinality of X is *m* and cardinality of Y is *n* then G is called a complete bipartite graph if for any vertex $u \in X$ and a vertex $v \in Y$, there is an edge *e* whose one end vertex is *u* and other is *v*. It is denoted by $K_{m, n}$. The number of edges in this graph is equal to *mn*.

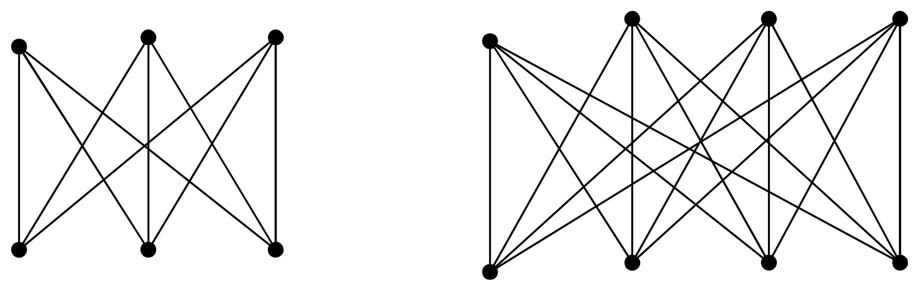

Fig. 1.11: Complete bipartite graph

Definition 1.28: Semiregular graph

A bipartite graph G is said to be semiregular if each vertex in the same partite set are of degree same.

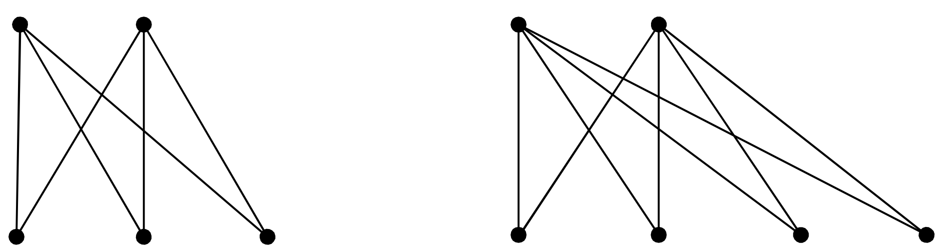

Fig. 1.12: Semiregular graph

Definition 1.29: Tree

A tree is a connected graph without any cycles. Obviously tree cannot have self loops or parallel edges.

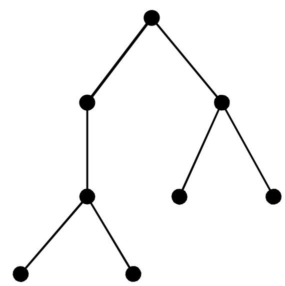

Fig. 1.13: Tree with eight vertices

Definition 1.30: Star graph

A graph G which contains no path of length three or more is called star graph. In other words, graph in the shape of stars is called star graph. A star graph $K_{1,n}$ is a tree of order $n+1$ consisting of one vertex adjacent to all others. Fig. 1.14 represents the star graphs with four and five vertices.

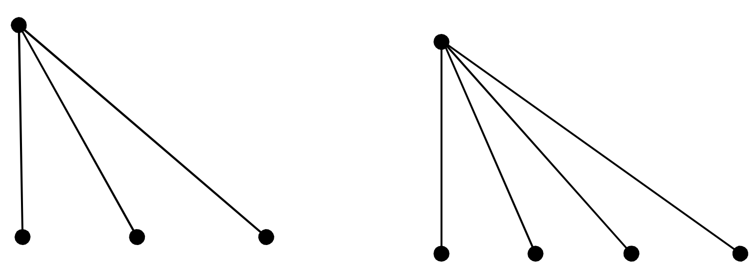

Fig. 1.14: Star graphs

Definition 1.31: Cyclic graph

A graph G is called cyclic if number of vertices in the graph is equal to the number of edges in the graph, as well as all the vertices can be arranged in a circle in such a way that two vertices are adjacent if and only if they are consecutive in the circle. It is denoted by C_n, cycle with n vertices.

For example the Fig. 1.15 represent the cyclic graph C_3 with three vertices.

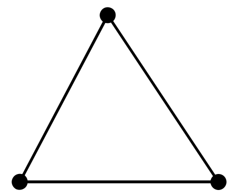

Fig. 1.15: Cyclic graph

Definition 1.32: Subgraph

Let G = (V, E) be a graph. A graph H = (V', E') is called a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. The Fig 1.16 is a subgraph of the Fig. 1.1 with three vertices and three edges as shown below:

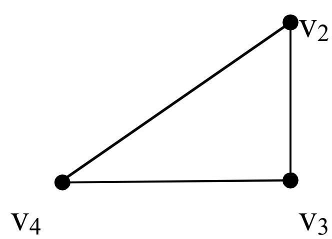

Fig.1.16: Subgraph of the graph in Fig. 1.1

Definition 1.33: Spanning subgraph

A subgraph H = (V', E') is called spanning subgraph of the graph G if V'=V.

Definition 1.34: Induced subgraph

A subgraph H = (V', E') is called an induced subgraph of the original graph G if each edge in G joining two vertices of H in H. For example the Fig. 1.16 is also induced subgraph of the graph G in Fig.1.1.

Definition 1.35: Proper Coloring

Suppose G is a graph. A coloring of the vertices of G in such a way that no adjacent vertices get the same color is called a proper coloring of G.

Definition 1.36: Chromatic number

Minimum number of colors required for proper coloring of a graph is called chromatic number.

Definition 1.37: Walk

Let G = (V, E) be a graph and u, v$\in$V. Then a walk in G from u to v is a finite alternating sequence v_0(= u)e_1v_1……$v_{k-1}e_kv_k$(= v) of vertices and edges such that $e_i = \{v_{i-1}, v_i\}$for i=1,2,3,……,k. The number k is called the length of the walk.

Definition 1.38: Path

A walk in which all the vertices are distinct is called a path. The number of edges present in such type of path is called length of the path.

Definition 1.39: Trail

A walk in which all the edges are distinct is called trail.

Definition 1.40: Distance

Let G = (V, E) be a connected graph and u, v $\in$V. Then the length of the shortest path from u to v is called length from u to v. It is denoted by d(u, v) or d_G(u, v).

Definition 1.41: Eccentricity

Let G be a connected graph and $u \in V$, then the eccentricity of u is defined by

$$E(u) = \max_{\substack{v \in V \\ v \neq u}} d(u,v).$$

Definition 1.42: Radius

The minimum of the eccentricities is called the radius of the graph G i.e.

$$R(G) = \min_{u \in V} E(u).$$

Definition 1.43: Central vertices

A vertex of minimum eccentricity is called central vertex.

If a graph contain more than one central vertex then that vertices are called central vertices.

Definition 1.44: Pigeonhole Principle

A well known proof technique in mathematics is the so-called pigeonhole principle, also known as the shoebox argument or Dirichlet drawer principle. In an informal way the pigeonhole principle says that if there are many pigeons and few pigeonholes then there must be some pigeonhole occupied by two or more pigeons.

We now give some basic theorems:

Theorem 1.1 [4]: THE HANDSHAKING THEOREM

If G = (V, E) be an undirected graph with *e* edges. Then

$$\sum_{v\in V}\deg(v) = 2e.$$

Proof: Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equals twice the number of edges. □

Theorem 1.2 [4]: In a non directed graph, the total number of odd degree vertices is even.

Proof: Let G = (V, E) a non directed graph.

Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.

Then

$$\sum_{v_i\in V}\deg(v_i) = \sum_{v_i\in U}\deg(v_i) + \sum_{v_i\in W}\deg(v_i)$$

$$\Rightarrow 2e - \sum_{v_i\in U}\deg(v_i) = \sum_{v_i\in W}\deg(v_i)$$

Now $\sum_{v_i \in W} \deg(v_i)$ is also even.

Therefore, $\sum_{v_i \in W} \deg(v_i)$ is even.

Thus the number of vertices of odd degree in G is even. □

Theorem 1.3: If G is a finite simple graph then there exist at least two vertices whose degree are same.

Proof: Let G = (*V*, *E*) be a graph with n vertices i.e. $V = \{v_1, v_2, \cdots, v_n\}$ then degree of each vertex $0 \le \deg_G(v_i) \le n-1$ because G is simple. If G contains a vertex of degree zero then degree of remaining vertices is at most $n-2$.

Thus if G has a vertex of degree zero then $0 \le \deg_G(v_i) \le n-2, \quad \forall v_i \in V(G)$.

Since there is vertices each of which degrees lies between 0 to *n-1* hence by pigeonhole principle, at least two vertices must have the same degree.

If each vertex of G has degree $1 \le \deg_G(v_i) \le n-1$ for each $i = 1,2,\cdots,n-1$ since *n* vertices get degree between 1 to *n-1* hence by pigeonhole principle two vertices must have the same degree. □

Theorem 1.4: A graph G is disconnected if and only if its vertex set *V* can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one vertex is in subset V_1 and the other is in subset V_2.

Proof: Let V_1 and V_2 be such a partition of the vertex set of G. Let *a* and *b* be any two arbitrary vertices of G such that $a \in V_1$ and $b \in V_2$.

Assume, G be connected. But there is no path between a and b, since, otherwise there would be at least one edge whose one end vertex is in V_1 and the other in V_2 which contradicts our assumption. Therefore G is disconnected.

Conversely: Let G be a disconnected graph. Let $a \in G$ and V_1 be the set of all vertices connected to the paths. Since G is disconnected, V_1 does not include all the vertices of G. The remaining such vertices from a set V_2 disjoint with V_1. Hence the partition. □

Theorem 1.5: A graph G is bipartite if and only if G has no cycle of odd length.

Proof: Let G be a bipartite graph with partite sets *X* and *Y*. Let C be a cycle and u be a vertex in C. Since, if we start through *u* along the cycle each time we reach to a vertex *v* which is in the same partite set as *u*, we travel through even number of edges and eventually we reach at *u* then C has even number of edges. This implies that G has no cycle of odd length.

Conversely: Suppose G has no cycle of odd length. We observe that G is bipartite graph if each non-trivial component is bipartite.

Let *H* be a non-trivial component of G and *u* be a vertex in *H*.

Let

$$X_H = \{v \in H : d(v,u) = \text{ even number}\}$$

and $Y_H = \{w \in H : d(w,u) = \text{ odd number}\}$.

Since *H* is connected, therefore $X_H \cup Y_H = V(H)$.

Since H is nontrivial component, therefore both X_H and Y_H are nonempty disjoint sets. That is there is no edge in *H* whose both end vertices are in (X_H or Y_H) same partite set.

If possible, an edge e whose end vertices are w_1 and w_2 both are in Y_H. Then there is a path from u to w_1 of odd length and also there is path from u to w_2 of odd length.

If we join these two paths with an edge e, then we get a cycle of odd length, by hypothesis which is contradiction. Hence H is bipartite.

Since H is arbitrary nontrivial component of G, this implies that each nontrivial component of G is bipartite.

Hence G is bipartite graph. □

APPLICATIONS OF GRAPH THEORY

1. Graphs in Markovian process [13]

Markov analysis is a way of analyzing the current movement of some variable. As a management tool, Markov analysis has been used during the last several years, mainly as a marketing aid for examining and predicting the behavior of customers from the stand point of the loyalty to one brand and their switching patterns to other brands.

The transition probability p_{ij} is the probability that the system presently in state E_i will be in state E_j at some later step. Whenever a new result or outcome occurs, process is said to have incremented one step. Each step represents a time period or any other condition which would result in another state.

A state-transition matrix is a rectangular array which summarizes the transition probabilities for a given Markov process. In such a matrix the rows identify the current state of the system being studied and column identify the alternative states to which the system can move.

For example, if E_i is the ith state of a stochastic process and p_{ij} is the transition probability of moving from state E_i to the state E_j in one step.
Then, a one stage state-transition matrix P is given below.

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ p_{31} & 0 & p_{33} \end{bmatrix} \end{array}$$

Table 1.1: Transition matrix

In the transition matrix of the Markovian chain, $p_{ij} = 0$, when no transition occur from ith state to *jth* state; and $p_{ij} = 1$, when the system is in the *ith* state and can move only to *jth* state at the next transition.
Each row of the transition matrix represents a one-step transition probability distribution over all state that is

$$p_{i1} + p_{i2} + \text{------} + p_{im} = 1, \ \forall i \quad \text{and } 0 \leq p_{ij} \leq 1$$

A diagram of the state transition matrix is called transition diagraph or weighted diagraph or transition diagram.
A transition diagraph shows the transition probabilities that can occur in any situation.

The arrow in the diagraph indicates the possible states to which a process can move from the given state. Such a diagraph is given in Fig. 1.17 and corresponds to the transition matrix in Table 1.1.

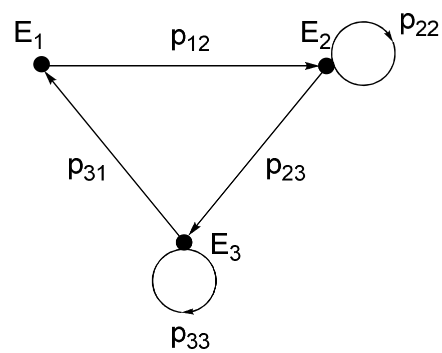

Fig. 1.17: Transition diagraph

2. Graphs in Network Analysis [10]

Network Scheduling is a technique used for planning and scheduling large projects in the field of construction, maintenance, fabrication, purchasing, computer system installation, research and development designs etc. The technique is a method of minimizing trouble spots, such as production bottlenecks, delays and interruptions, by determining critical factors and coordinating various parts of the overall job.

A network is a graphic representation of a project operations and is composition of activities and events. Activity is represented by an arrow whose vertices are called events.

3. Graphs in Biological Science

A recent application of graphs is in biological science. Here diagraphs, cyclic graph and hypergraphs are used to represent biological phenomena. The nodes of the digraph represent entities while edges represent biological interactions between the nodes such as transformation, catalysis, complex function or any other biological process.

4. Graphs in Groups

A group of order *n* can be represented by a strongly connected diagraph of *n* vertices, in which each vertex corresponds to a group element and edges carry the label of a generator of the group. Thus the graph of a cyclic group of order *n* is a directed circuit of *n* vertices in which every edges has the same label. The diagraph of a group uniquely defines the group by specifying how every product of elements corresponds to a directed edge sequence. This diagraph, known as the Cayley diagram, is useful in visualizing and studying abstract groups. Coding theory uses groups and as such graph theory plays an important role in it.

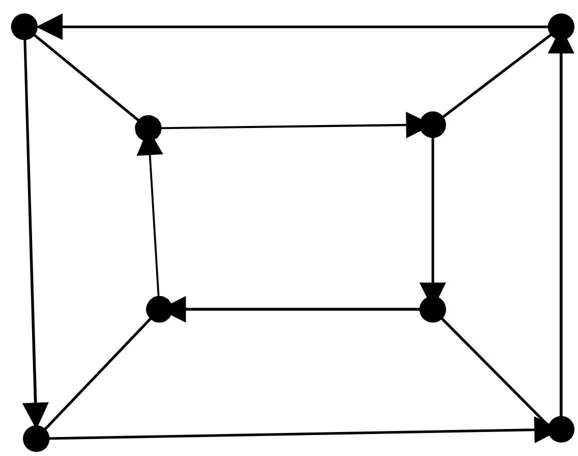

Fig. 1.18: Cayley diagram