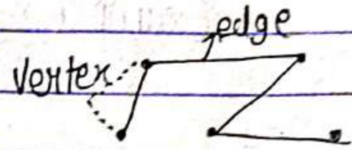
Graph theory

Definition :- Collection of order pair Node/vertex (V) and edge (E)



$G = \{ (V, E) \}$

such that

⇒ V is finite set of vertices

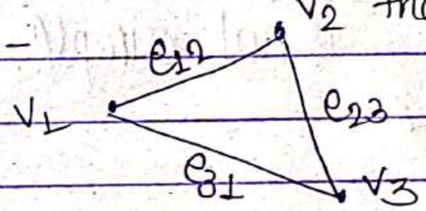
⇒ E is finite set of edges

Example ① $V = \{ v_1, v_2, v_3 \}$

$E = \{ e_{12}, e_{23}, e_{31} \}$

thats mean 1 → 2 vertex

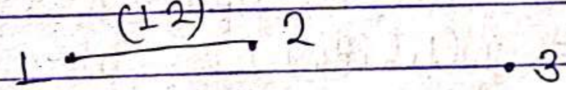
graph –



② $V = \{1, 2, 3\}$

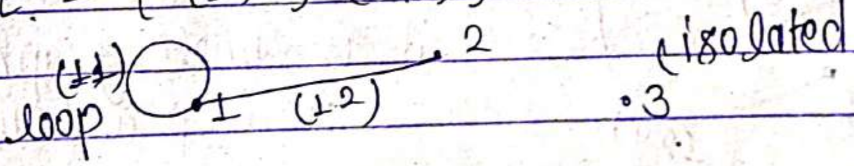
$E = \{ (1\,2) \}$

graph



③ $V = \{1, 2, 3\}$

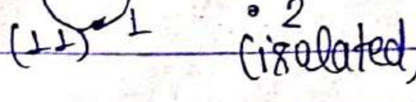
$E = \{ (1, 2)\ (1, 1) \}$



Loop definition → If starting and ending node is same then edge is known as loop

Ex = V = (1, 2)

E = (1, 1)

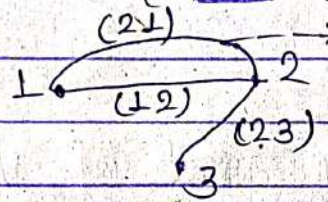


Date:

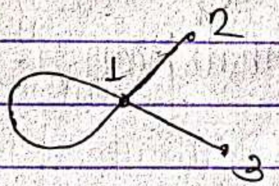
Page:

Multiple edge ⇒ Relation of two node / vertex more than 1.

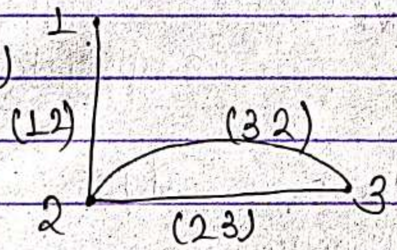
Example $V = \{1, 2, 3\}$
$E = \{(12)\ (23)\ (21)\}$



Simple graph ⇒ A graph G is said to be simple if
(i) It has no loop.
(ii) It has no multiple edges.

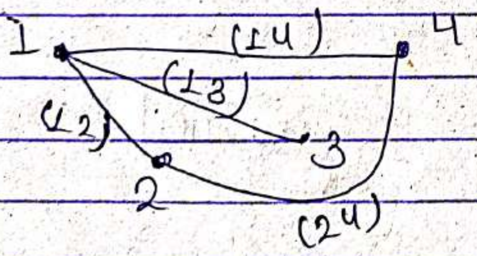
Example (i) G = 

is not simple

(ii) 

is not simple

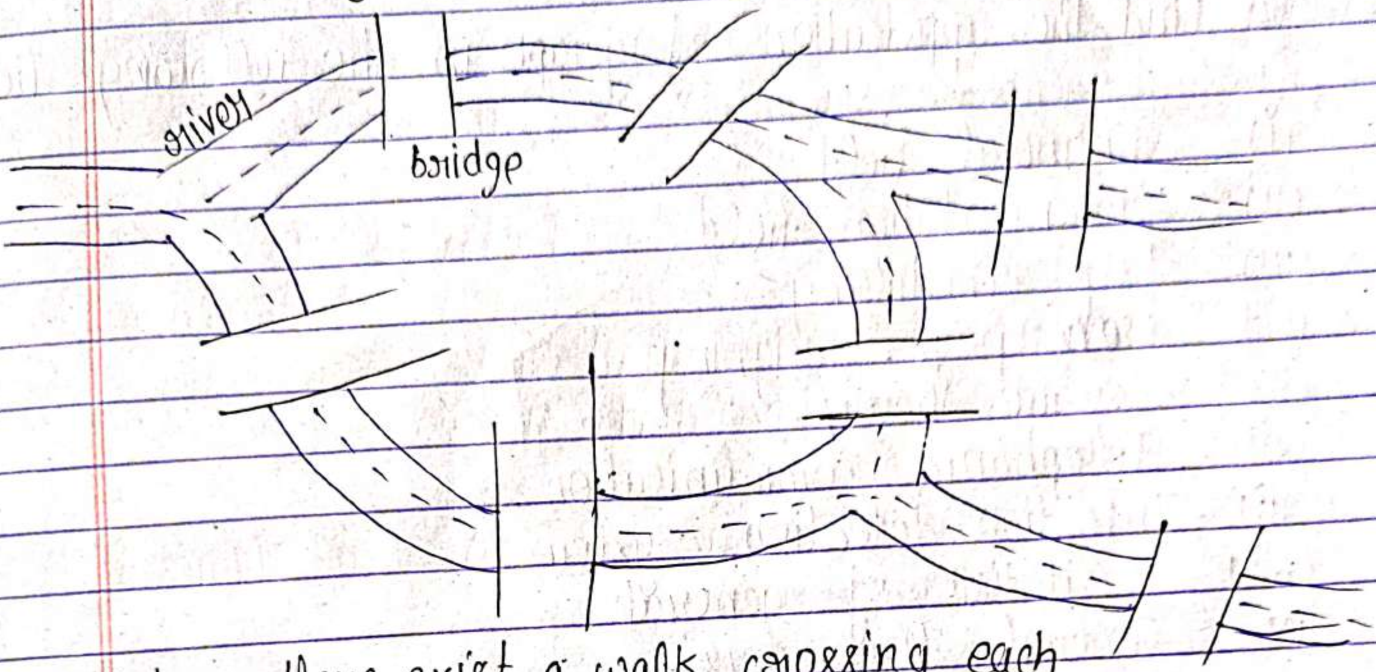
(iii) $V = \{(1, 2, 3, 4\}$
$E = \{(12)\ (13)\ (24)\ (14)\}$



it is simple graph
no loop and no multiple edges

Date:
Page:

Knoingsbery bridge problem



does there exist a walk crossing each of the seven bridges exactly end.

~~solution~~

Graph :- A graph G is an ordered pair of G = (V,E)

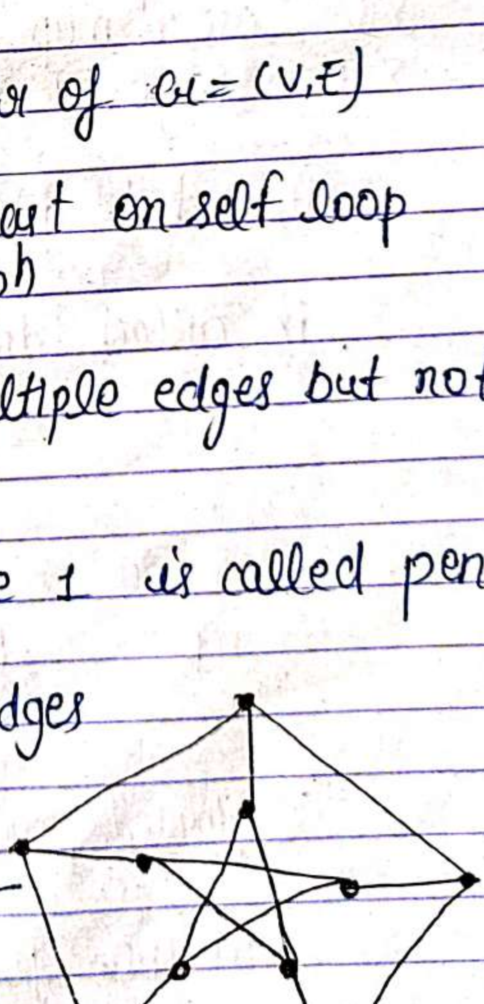
Psuedo graph :- A graph which contains atleast on self loop is called pseudo graph

Multiple ~~ed~~graph :- which contain multiple edges but not self loop

Pendant vertex : a vertex of degree 1 is called pendant vertex

Peterson graph : each vertex has 3 edges

Peterson graph of 10 vertices ←



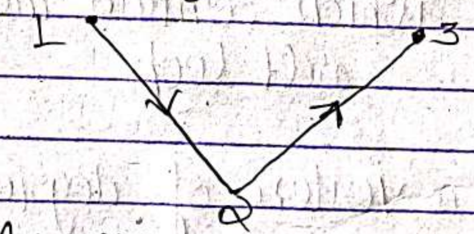
Date:
Page:

Assignment - I

Find the application of graph in the following fields

(i) financial field
(ii) Transpotation field
(iii) Task scheduling
(iv) web page designing
(v) Game theory
(vi) Telephonic communication
(vii) Infection detection system
(viii) Citation of general
IX controal flow system
X computer network
XI Shortest root finding
XII road map
XIII flyover map

Directed graph :- a graph where each edge has direction associated to it is called directed graph.



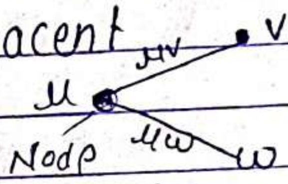
Example = fly over map
one way route

Order of graph G : No. of vertices in graph G is called order of G.
and no. of edges is called size of G.

Date:
Page:

Adjacent vertices :- if u and v are ends of edge E such that e = u-v then u and v are adjacent edges

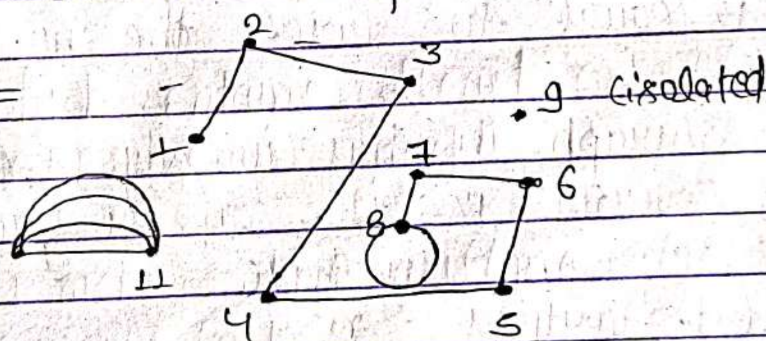
Adjacent edges :- If $e_1 = uv$ $e_2 = uw$ then they have a common end node u so e_1 and e_2 are adjacent edges



Disconnected graph :- if ∃ an isolated node and vertex in G then G is called disconnected graph

Degree of vertices in graph :- degree of vertices is the no. of edge incident to it with loops counted twice.

Example =



V = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
E = {(12), (23), (34), (45) (56) (67) (78) (88) (10 11)}

degree of vertex 8 = 3 {loop counted is twice}

degree of vertex 9 = 0

degree of vertex 2 = 2 (that means 2 edges)

degree of vertex 10 = 4

degree means – No. of edge in a point

Date:

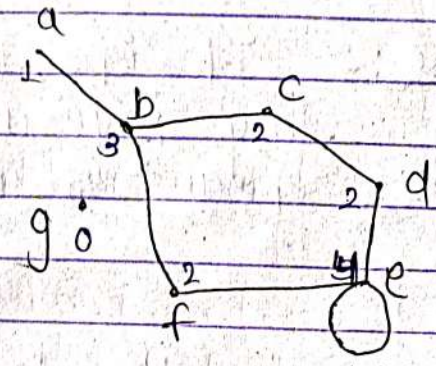
Page:

Notation ↓

Maximum degree of ~~vertex~~ Graph $= (\Delta G)$

Minimum degree of Graph $= (\delta G)$

Example



maximum deg of graph = 4

minimum deg of graph = 0

Hand - shaking lemma

<u>Degree sum formula</u> :- The sum of all degrees of vertices in a graph G is equal to twice the no. of edges.

Ex ① Graph G has 10 edges and degree of each vertex is 4. can u guess how many vertices graph G have

let vertices = n

$$\sum d(v) = 2 \times \text{edge}$$

$$4n = 2 \times 10$$

$$n = \frac{20}{4} = 5$$

② Graph H claimes that it has 7 edges and each vertex has 3 degree. do u think graph H is telling truth or not

Not

Date:
Page:

$$\Sigma d(V) = 2 \times edges$$
$$3n = 2 \times 7$$
$$n = \frac{14}{3}$$

its not possiple because vertices has no rational form

20/07/23

Isomorphic Graph

two graph G_1 and G_2 are said to isomorphic graph if $\exists$ a bijection b/w G_1 and G_2

$$G_1 \equiv G_2$$

Conditions :-

If two graph G_1 and G_2 are isomorphic

(a) $|V(G_1)| = |V(G_2)|$

No. of vertices in G_1 = No. of vertices in G_2

(b) $|E(G_1)| = |E(G_2)|$

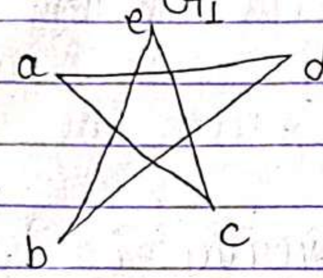
No. of edges in G_1 = No. of edges in G_2.

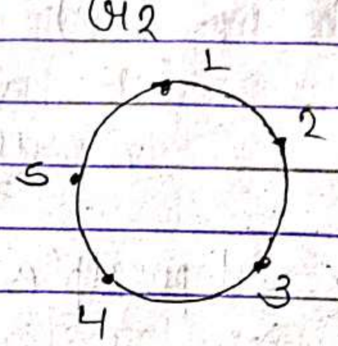
(c) degree sequence of G_1 and G_2 is same.

(d) If the vertices $\{v_1, v_2, --- v_K\}$ form a cycle of length K in G_1.

then $\{f(v_1), f(v_2), --- f(v_K)\}$ form a cycle of same length K in G_2

Example





determine whether G_1 and G_2 are isomorphic Graph

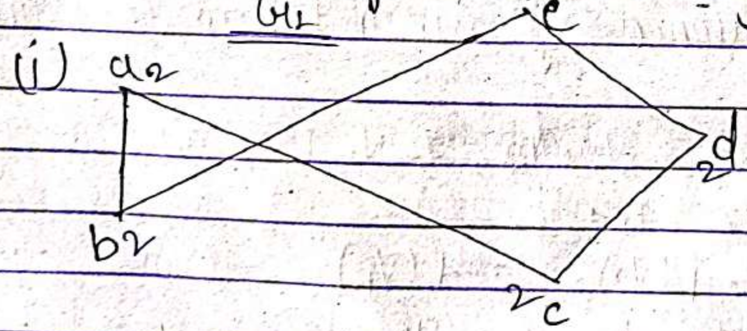
$V(G_1) \; \{a, b, c, d, e\}$
$V(G_2) = \{1, 2, 3, 4, 5\}$

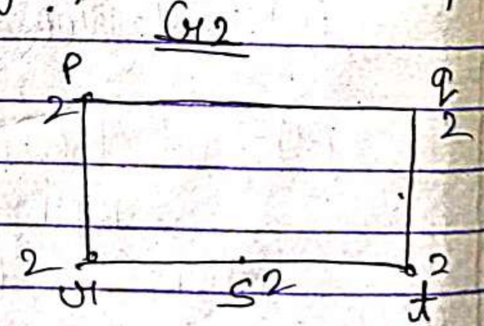
$f : V(G_1) \rightarrow V(G_2)$

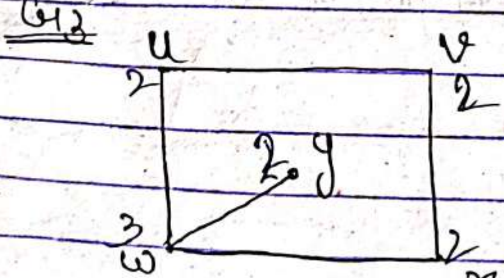
a — 1	(ad) - (1 2)
b - 3	(ac) - (1 5)
c - 5	(eb) = (4 3)
d - 2	(ec) = (4 5)
e - 4	(da) = (2 1)
	(d b) = (2 3)

Example :- which of the following graphs are isomorphic

(i)





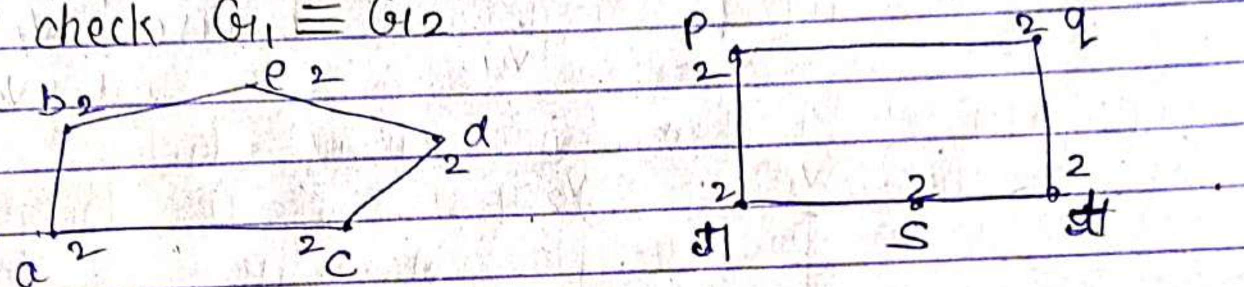


(1) $G_2 \not\equiv G_3$
G_2 is not isomorphic in G_3
since degree sequence of

G_2 are not match G_3

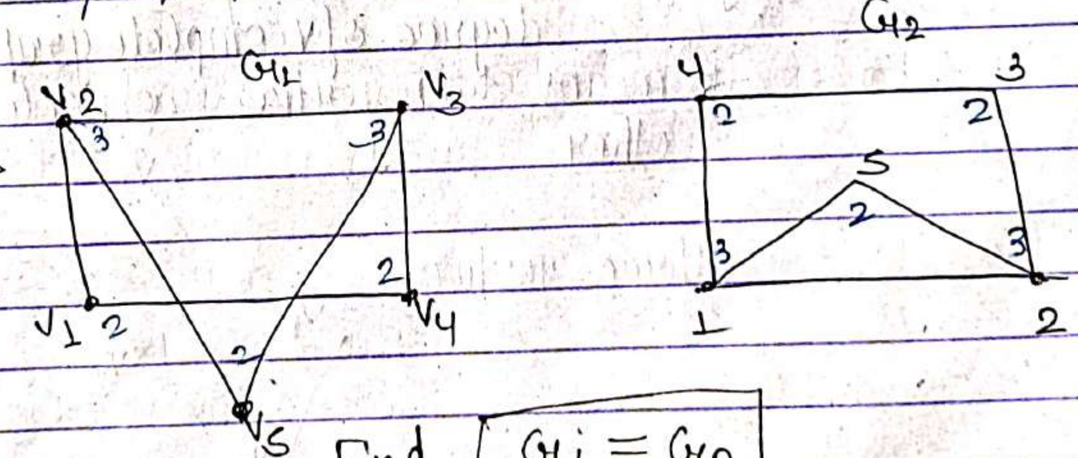
(2) $G_1 \not\equiv G_3$ (G_1 & G_3 are not isomorphic) because degree sequence of G_1 are not match G_3 }

(3) check $G_1 \equiv G_2$



a - ~~t~~ r	ac - ~~p~~ rs
b - ~~q~~ p	cd - st
c - ~~r~~ s	de - tq
d - ~~s~~ t	eb - pq
e - ~~t~~ q	ba - rp

Example ⇒



Find $\boxed{G_1 \equiv G_2}$

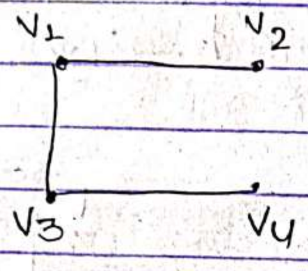
$V_1 - 3$	$V_1V_2 = (3\ 2)$	$V_2V_5 = (2\ 5)$
$V_2 - 2$	$V_2V_3 = (2\ 1)$	$V_1V_4 = (3\ 4)$
$V_3 - 1$	$V_3V_4 = (1\ 4)$	
$V_4 - 4$	$V_3V_5 = (1\ 5)$	therefore $G_1 \equiv G_2$
$V_5 - 5$		G_1 isomorphic G_2

Date: 24/07/23
Page:

No. of edge $\Rightarrow {}^nC_2$

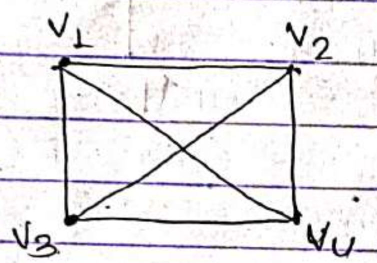
(K_n) Complete Graph :- graph G is said to be complete graph if for every pair $u, v \in V$ $\exists$ edge $uv \in E$

Example = ①



is not complete graph because $v_2, v_4 \in V$ but $(v_2 v_4) \notin E$

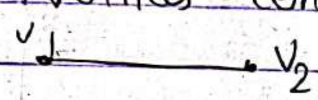
②



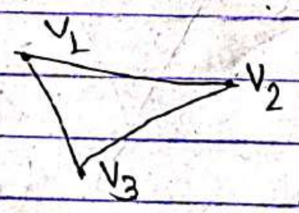
If $\exists$ at least one pair $u, v \in V$ such that edges $uv \notin E$ then G is not complete graph

complete graph is denoted by K_n (n = no of vertices)
degree of complete graph is $(n-1)$

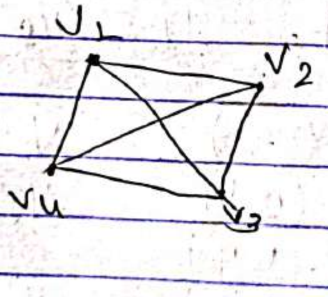
K_2 = its means two vertices connected with each other



K_3 = three vertices

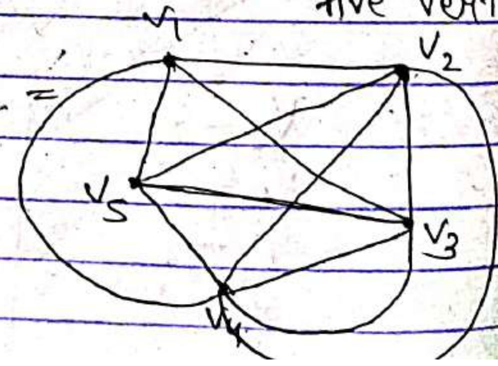


K_4 =



five vertices

K_5 =

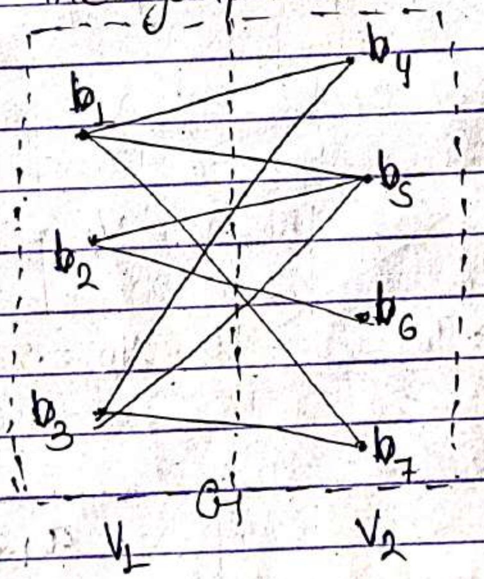


Date:
Page:

Regular graph - A graph G is said to be regular graph if every vertex has degree K and it is also a complete graph

Bipartite graph - a graph $G = (V, E)$ is said to be bipartite graph if $\exists$ a partition of V in V_1 and V_2 such that
(i) $V_1 \cup V_2 = V$
(ii) $V_1 \cap V_2 = \phi$
(iii) no two vertices of V_1 are adjecent
(iv) no two vertices of V_2 are adjacent
denoted by $K_{m,n}$ (is said to complete if $m = n$)

Ex ⇒ Determine the graph G is bipartite or not



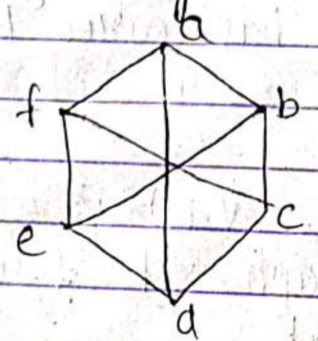
(i) $V_1 \cup V_2 = V$
(ii) $V_1 \cap V_2 = \phi$
(iii) No two vertices of V_1 are adajecent.
(iv) No two vertices of V_2 are adjacent.

So, its bipartite graph

Date:
Page:

Que Which of the following graph is bipartite graph

a)



$V_1 = \{a, c, e\}$ ∵ no connected each other

$V_2 = \{b, d, f\}$

(i) $V_1 \cup V_2 = V$

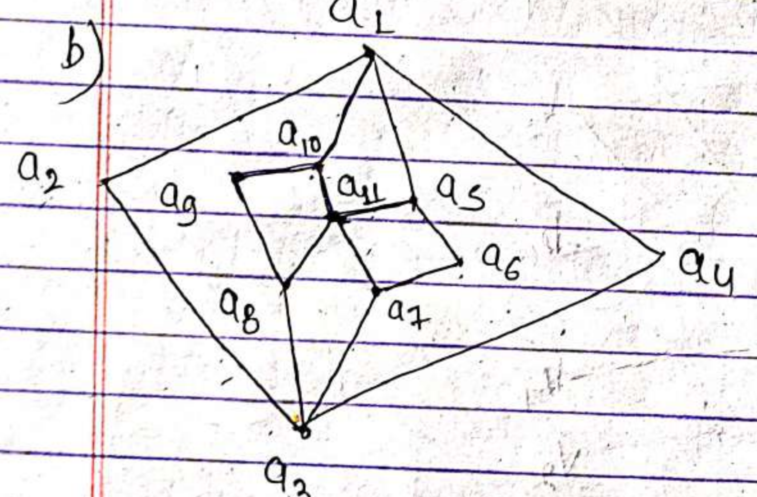
(ii) $V_1 \cap V_2 = \phi$

(iii) no two vertices of V_1 are adjacent

(iv) no two vertices of V_2 are adjacent

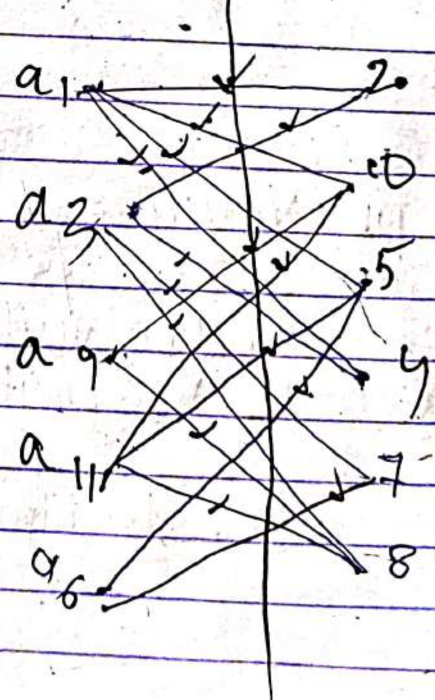
so it's bipartite graph

b)



$V_1 = \{a_1, a_6, \cancel{a_7}, a_3, \cancel{a_8}, a_9, a_{11}\}$

$V_2 = \{a_4, a_2, a_7, a_8, a_5, a_{10}$



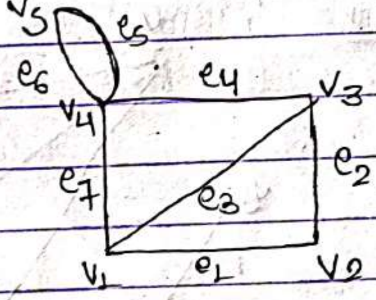
Date: 25/07/23
Page:

Matrix representation of Graph - There are two ways to write matrix representation of graph (to store graph in computer)

(1) Incidence matrix ⇒ Let G be a graph $G = (V, E)$ such that $V = \{v_1, v_2, \ldots v_n\}$ and $E = \{e_1, e_2, \ldots e_m\}$ provided that graph has no self loop, then incidence matrix $[a_{ij}]_{n \times m}$ is defined as -

$$a_{ij} = \begin{cases} 1, & \text{if } e_j \text{ edge is incident to } v_i \text{ vertex} \\ 0 & \text{otherwise} \end{cases}$$

Example Write incidence matrix of following graph

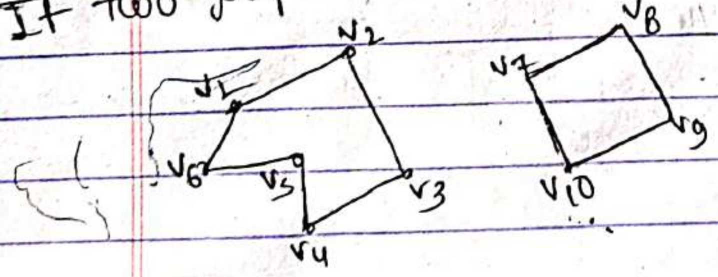


$$\begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}_{5 \times 7}$$

$|V(G)| = 5$
$|E(G)| = 7$

Note:- (1) sum of enteries in i^{th} row in incidence matrix
(2) To know each edge is incident to how many vertex
(3) If how many row of the matrix is zero then the corresponding vertex is isolated

If two graph which are not connected but ~~not~~ count in one graph

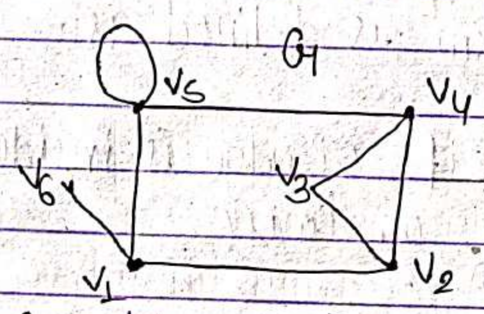


$$\left[\begin{array}{c|c} I(G_1) & 0\ 0\ 0 \\ & 0\ 0\ 0 \\ \hline 0\ 0\ 0\ 0 & I(G_2) \\ 0\ 0\ 0 & \end{array}\right]$$

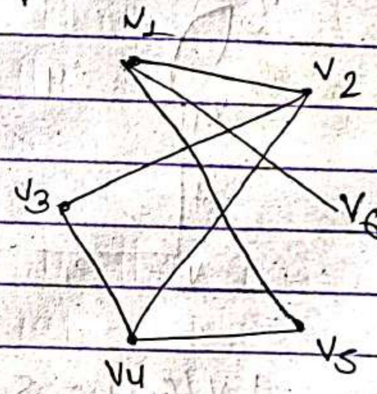
(2) Adjacency Matrix :- Let $G = (V, E)$ be a graph such that $|V| = n$, provided that no parallel / multi edges are allowed while loops are allowed

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is adjacent to } v_i \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \begin{bmatrix} & \\ & \end{bmatrix}_{v_1 \text{---} v_n}$$

Example :-



① Diagonal entry shows self loop
② Always a symmetric matrix

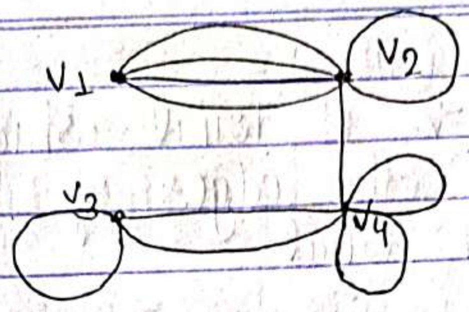


$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1^{\text{loop}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

Matrix associated to Multigraph :- Multigraph is a graph where multiedges as well as self loop are allowed.

$$a_{ij} = \begin{cases} r, & \text{if there are } r \text{ no. of edges b/w vertices} \\ 0 & \text{otherwise} \end{cases}$$

Date:
Page:

Example :- 

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 4 & 0 & 0 \\ v_2 & 4 & 1 & 0 & 1 \\ v_3 & 0 & 0 & 1 & 2 \\ v_4 & 0 & 1 & 2 & 2 \end{array}$$

Walk = let $G = (V, E)$ be a graph. A walk in G is an alternating sequence of vertices and edges starting and terminating with a vertex such that each edge e_i is incident on the vertices which proceed and succede it. i.e., $W: v_1\ e_1\ v_2\ e_2\ v_3\ e_3 \text{---} v_{n-1}\ e_{n-1}\ v_n$

length of walk - No. of edges in the walk is called length of walk

Trail ⇒ A walk in ways of the edges are distinct is called a Trail

OR No edges repeated but vertex repeated

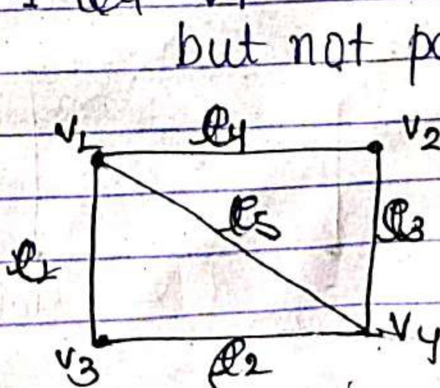
Path ⇒ A walk in which all the vertices are distinct is called a path

OR neither vertex nor edges repeated

Ex $W: v_1 e_1\ v_2 e_2\ v_3\ e_5\ v_1\ e_1\ v_2$ ---
but not trial

$T: v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_1\ e_4\ v_4$
but not path

$P: v_1\ e_1\ v_2\ v_3\ e_3\ v_4$



Eularian trial ⇒ A Trial which contains all the edges of the graph is called eularian trial.

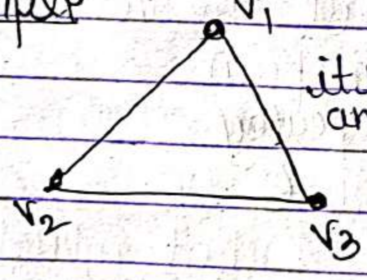
Euler graph ⇒ A graph which contain closed eularian trial is called euler graph

Hamiltonian path ⇒ A path which contain all the vertices of the graph is called Hamiltonian graph

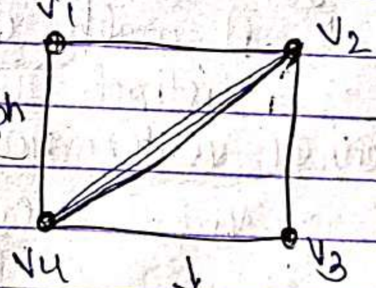
Hamiltonian cycle or circle ⇒ A cycle that contains all the vertices of the graph is called Hamiltonian.

Hamiltonian graph ⇒ A graph that has a hamiltonian cycle is called Hamiltonian graph.

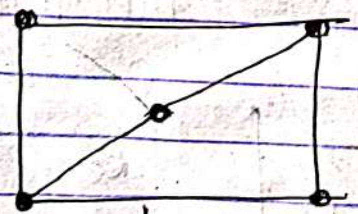
Example



it is euler and hamilton graph both



Not euler but hamiltonian



Neither euler nor Hamiltonian