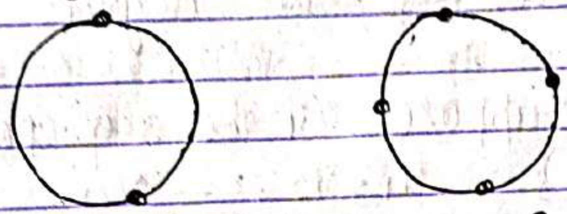
Date: 9|08|23

Page:

Cyclic graph :- denoted by C_n

a graph $C_n = (V, E)$ ~~satisfy~~ is called cyclic if it satisfying $|V| = n$ & $|E| = n$ and the vertices can be arranged in a circle in such way that two vertices are adjacent in C_n iff they are consecutive in the circle.

Example



C_2 $\quad$ C_4

Remark :- C_n is simple or $n \geq 3$

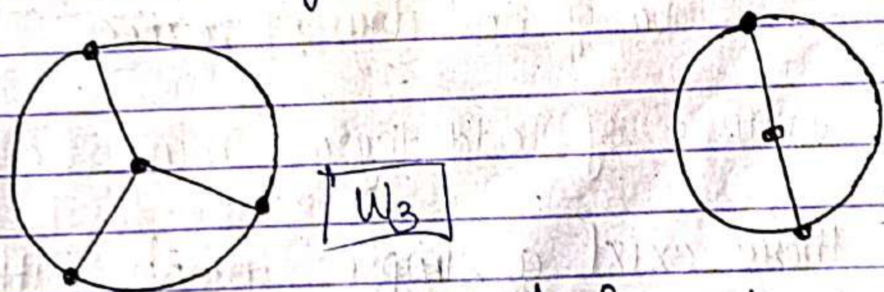
C_2 is not simple.

Wheel graph :- (W_n) a graph $G = (V, E)$ is called wheel graph if $G = C_n \cup \{v_{n+1}\} = W_n$

where,

v_{n+1} adjacent to all the other vertices in C_n

Ex ⇒



$\boxed{W_3}$ $\quad$ $\boxed{W_2}$

In wheel graph,

No. of vertex $|V| = n+1$

No. of Edges $|E| = 2n$

Date:
Page:

Hand Shaking Lemma :-

Proof since each edges contributes to the two degree sum.

Hence ~~sum~~ : $\sum_{i=1}^{n} deg(v_i) = 2 \times$ No. of edges
$= 2|E|$

Corallary :- In any finite graph $G = (V, E)$ the no. of vertices of odd degree is an even number

Proof suppose G be a graph with n vertices i.e; $v_1, v_2, --- v_n$ then.

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

let, $$\sum_{i=1}^{r} deg(v_i) + \sum_{i=r+1}^{n} deg(v_i) = \text{even}$$

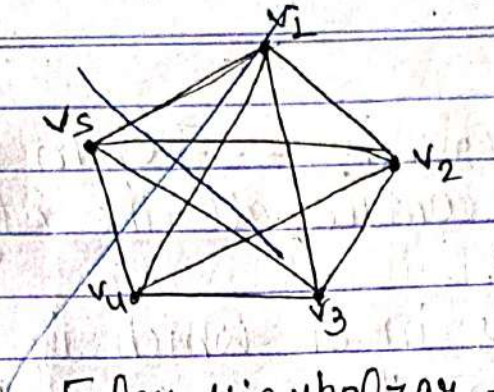
$$\text{even} + \sum_{i=r+1}^{n} deg(v_i) = \text{even}$$

$$\Rightarrow \underbrace{\sum_{i=r+1}^{n} deg(v_i)}_{\text{sum of odd degree vertices}} = \text{even}$$

$\Rightarrow$ No. of vertices of odd degree must be an even number

Qus does there exist a simple graph with 5 vertices of degree 1, 2, 3, 4, 5 ?
If Yes draw such a graph.
If No. given appropriate reason.

Date:
Page:



Euler Hierholzer theorem

a connected graph is an euler graph iff every vertex has even degree

Proof :- suppose G is an euler graph. let v be an vertex in G. ~~let w be a closed~~
then $\exists$ a closed eulerian trial in G. let it be w.
suppose v is repeated in w k times.
since each time we passes through the vertex v in the trial.
we come out of v through an edge. hence degree of vertex

$$\deg_w(v) = 2K$$

since all the edges in G are in w. Hence degree of the vertex

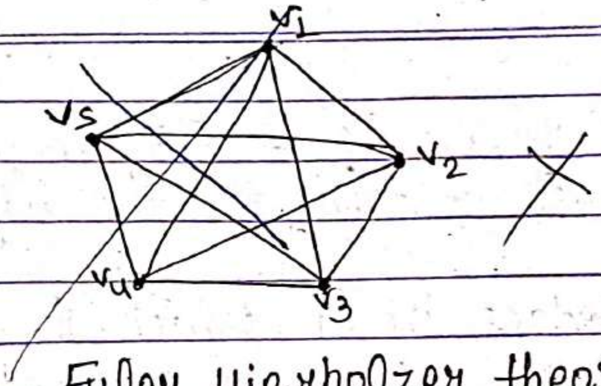
$$\deg_G(v) = 2K$$

degree of non repeated vertex $K = 1$

$$\deg(v) = 2$$

$\Rightarrow$ every vertex has even degree in euler graph

Conversely, suppose G is a connected graph $\ni$: degree of vertex $\deg_G(v)$ is even for each $v \in G$



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⇒ every vertex has even degree in euler graph

conversely, suppose G is a connected graph. ∋: degree of vertex $deg_G(v)$ is even for each $v \in G$

Date:
Page:

let, $\omega : v_0 e_1 v_2 e_2 v_3, e_3 --- e_n v_n$ be largest trail in G. must contain all the edges adjacent to v_n.
(If some edges in G which is incident to v_n is not in ω then ω can not be enlarged by adding that edges)
since degree of v_n is even.
Hence; $v_0 = v_n$
then ω is a closed trail.
further, we claim that, $E(G) = \{e_1, e_2, ---, e_n\}$
if $\exists$ $e \in G$ $\ni$: e is not in ω.
then, $e = \{u, v_i\}$ $1 \leq i \leq n$

let, $\omega_1 : u e_i v_i e_{i+1} --- e_n v_n e_1 v_1 e_2 v_2 --- e_i v_i$
then ω_1 is a walk whose length $= n+1 > e(\omega)$
which is contradiction
Thus $E(G) = \{e_1, e_2, --- e_n\}$
$\Rightarrow$ ω is an euler trail.
$\Rightarrow$ G is an euler graph.

Pigeon hole principle

The pigeonhole principle says that if there are many pigeon and few pigeon hole then there must be some pigeonhole occupied by two or more pigeons

Result
Ques If G is a finite simple graph then there exist atleast two vertices whose degree are same.

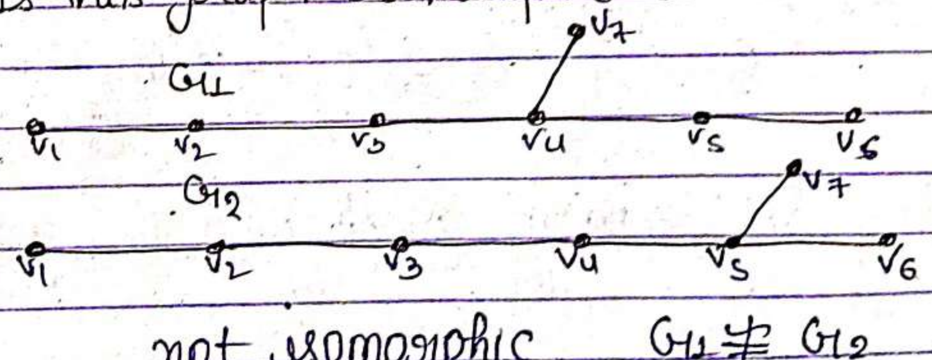
Proof

~~If G is finite~~

Proof Let $G = (V, E)$ be a graph with n vertices
i.e; $V = \{v_1, v_2, \ldots, v_n\}$
the degree of each vertex $0 \le \deg(v_i) \le n-1$
because G is simple. If G contains a vertex
of degree 0, then the degree of remainning
vertices is atmost $n-2$
thus, if G has a vertex of degree 0
then, $0 \le \deg(v_i) \le n-2 \quad \forall\ v_i \in V(G)$
since, there are vertices each of which has
degree between 0 to $n-2$
Hence, by pigeon hole principle atleast two
vertices must have the same degree.

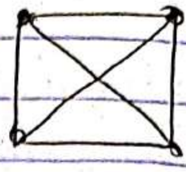
Case 2 :- If each vertex of G has degree ~~by~~ b/w
$1 \le \deg(v_i) \le n-1 \quad \forall\ v_i \in V(G)$
since each vertex get degree between 1 and $n-1$
hence by pigeon hole principle two verotices
must have the same degree.

Qus Is this graph isomorphic or not



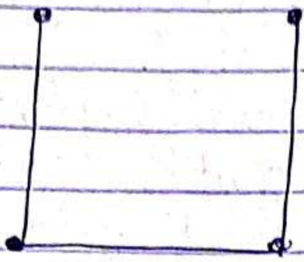
not isomorphic $G_1 \not\cong G_2$

because adjacency of v_4 in G_1 and v_5 in G_2
are not same

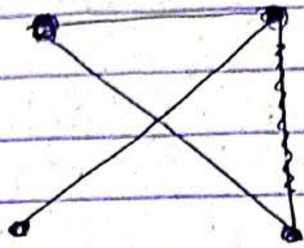


complementory graph :-

for ex



G_1



$\overline{G_1}$

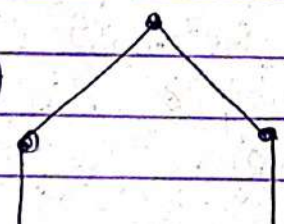
$\Rightarrow \overline{G_1}$ is the complement graph of G_1

(i) ↑ is the self complimentory graph of 4 vertices

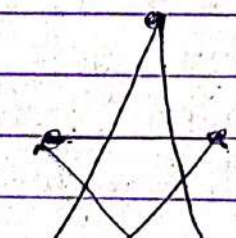
Self complimentory graph :- a graph G_1 is said to be self complim-graph if $G_1 \cong \overline{G_1}$

Qus Show that there does not exist a self complimentory graph with 6 vertex.

Ex (2)



G_1



$\overline{G_1}$

$G_1 \cong \overline{G_1}$ is self complimentory graph of 5 vertices

Tree ⇒ A connected graph with no cycles is called tree.

OR

i.e; [A corrected acyclic graph is called tree)

it is denoted by (T)

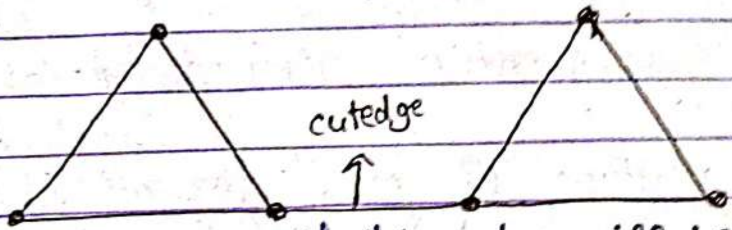
T is tree iff T−V is a tree for each pendant vertex V.

In tree No of edges $(n-1)$

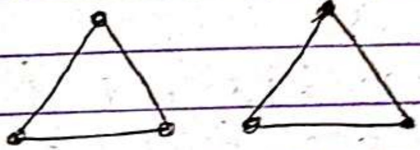
Definition (Cutedge or Bridge)

an edge 'e' of the graph G is called cutedge or bridge if G−e has more than one component i.e; graph is disconnected.

Ex ⇒



If this edge will be delete then this graph will be disconneted so this edge is called cutedge or bridge



Result :- Euler graph has no bridge

Proof suppose G is an euler graph than G is connected and every vertex in G is even degree.

by contradiction.

If possible, let e be a bridge in G, then G−e has two component G_1 & G_2

let V_i & V_j be end vertices of the edge e.

and let, $V_i \in G_1$
$V_j \in G_2$

then, all vertices in G_1 except V_i has even degree

i.e; no. of vertices of odd degree in $G_1 = V_1$

this is impossible
thus our hypothesis is not true

$\Rightarrow$ G has no cutedge or bridge

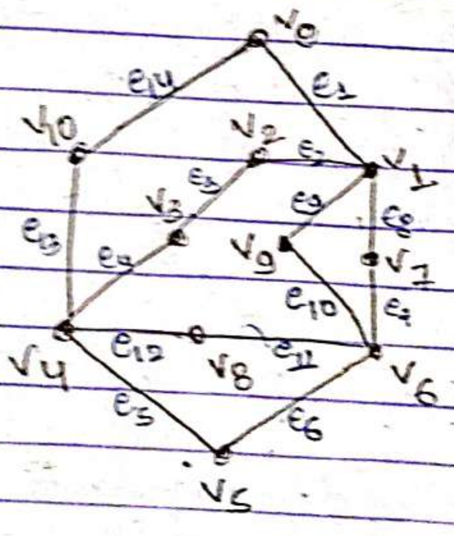
Algorithm for finding euler trial :- let G be a connected graph with each vertex of even degree and let
$E = \{e_1, e_2, \ldots e_m\}$

Step I : start with any vertex v_0 put $T_0 = \{v_0\}$

step II : If all the edge have been including then stop and the resulting trial is an euler trial otherwise go to step III

step III : let $T_i = \{v_0 e_1 v_1 \ldots e_i v_i\}$

$e_{i+1} \in G_i (V, E - \{e_1, e_2, \ldots, e_i\})$
$G_{i+1} (V, E - \{e_1, e_2, \ldots e_i, e_{i+1}\})$
$\vdots$
and so on



$T_0 = \{v_0\}$
$T_1 = \{v_0 e_1 v_1\}$
$T_2 = \{v_0 e_1 v_1 e_2 v_2\}$
$\vdots$
$T_{14} = \{v_0 e_1 v_1 e_2 v_2, \ldots e_{14} v_0\}$

Pendent vertex → a vertex has one degree In tree

Date:

Page:

Internel vertex → If vertex has mare than 1 degree intree

<u>Theorem</u> The following astatement are equivalent

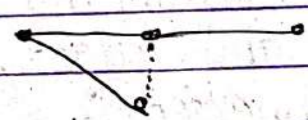
(i) Graph G is a tree

(ii) Between any two distinct vertices in G

(iii) there is a unique path in G.

→ G is ~~no~~ minimally conneted (i.e; (G − e) is disconneted) for each $e \in G$

(iv) G is maximal acyclic graph (i.e; (G + e) contain a cycle)



(v) G is acyclic & $|V| = |E| + 1$

<u>Proof</u> (i) ⟶ (ii)

suppose G be a tree. since G is a connected there is a path joinning any two distinct vertices in G.

If possible asuppose there are two distinct path P_1 and P_2 between the vertices v_0 & v_K in G asuppose v_i & v_j be present in both P_1 and P_2 and no vertex common in P_1 & P_2 b/w v_i & v_j let. c denote a walk along P_1 from v_i to v_j and P_2 from v_j to v_i. then c is a cycle containning v_i & v_j in G

This is a contradiction

Hence In G b/w any two vertices there is a unique path.

Date:
Page:

(ii) $\longrightarrow$ (iii)
let e be a any edge in G with end vertices v_i & v_j
then P: $v_i e v_j$ is the only path from v_i to v_j
$\Rightarrow$ G-e contained no path from v_i to v_j
$\Rightarrow$ G-e is disconnected
Hence, G is minimally connected

(iii) $\longrightarrow$ (iv)
let G is a minimally connected graph
$\Rightarrow$ G is connected and acyclic, because if G has a cycle containning v_i & v_j say c.
let e be any edge in the cycle then G-e is ~~s~~till connected.
$\Rightarrow$ G is not minimally connected
$\Rightarrow$ G is acyclic
we claim that G is maximal connected.
let e be any edge not in G with end vertices v_i & v_j.
G is connected hence there is a path P from v_i to v_j in G. then $P \cup e$ is a cycle in G+e
$\Rightarrow$ G is maximal acyclic graph

(iv) $\longrightarrow$ (v) by mathematical induction:
~~s~~uppose, G is maximal acyclic graph.
we have to prove that $|V| = |E|+1$
let, ~~since~~ G is connected and without a cycle.
we use mathematical induction on NO of vertices
$|V| = n = 1$
the result hold
~~s~~uppose the result hods for each connected acyclic graph with less than n vertices.

Date:
Page:

let G be a connected acyclic graph with n vertices and $e \in G$ be any edge. then $G-e$ has two components C_1 & C_2.
clearly C_1 & C_2 are acyclic with less than n vertices
suppose C_1 has K_1 vertices then C_2 has $n-K_1$ vertices
since C_1 has $K_1 < n$ vertices hence no. edges in
$C_1 = K_1 - 1$
similarly, No. of edges in $C_2 = n - K_1 - 1$
then No. of edges in $G-e = C_1 + C_2$

$$= K_1 - 1 + n - K_1 - 1$$
$$= n-2$$

then No. of edges in $G = (n-2)+1 = |n-1|$

$$|E| = |V| - 1$$

(V) $\rightarrow$ (I)

Date: 2/09/23
Page:

Definition :- let $G = (V, E)$ be a tree, a vertex $v \in G$ of degree > 1 is called an internal vertex.

Definition :- (i) A tree T is called m-nary tree if each vertex has degree $\leq m$

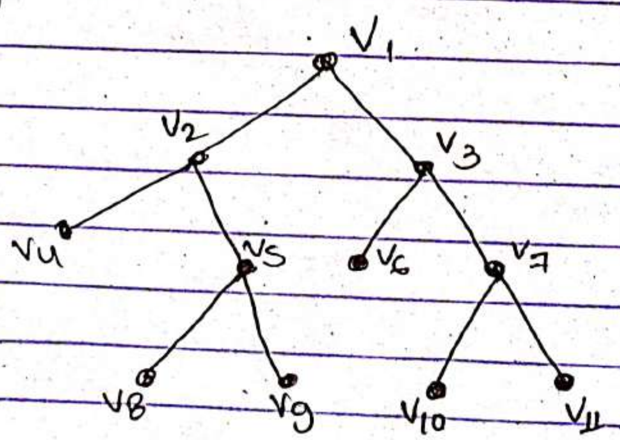
(ii) A m-nary tree is called a full m-nary tree if each internal vertex has degree m.

Rooted tree ⇒ A tree is called rooted if it has a distinguished vertex v_0 called root of the tree.

Rooted Binary tree ⇒ A Binary tree in which there is a vertex of degree 2 (called root) and all other internal vertices of degree 3 is called rooted binary tree.

Note:- A 3-nary tree is called binary tree if full binary tree has each internal vertex of degree.

Ex ⇒



Prove that

Ques If T is a rooted binary tree with n vertices then it has $\frac{n+1}{2}$ pendant vertices

Date:
Page:

Proof suppose T has p pendant vertices then degree sum of vertices in T

$$\therefore 2(n-1) = 2 + p \cdot 1 + 3(n-p-1)$$
$$2n-2 = 2 + p + 3n - 3p - 3$$
$$2p = 2 + 2 + 3n - 3 - 2n$$
$$2p = 4 + n - 3$$
$$\boxed{p = \frac{n+1}{2}}$$

Number of vertices of degree 3 $= n - \frac{n+1}{2} - 1$

$$= \frac{n-3}{2}$$

Assignment

1. Show that the number of vertices in a rooted binary tree is odd

2. Any tree with $n (\geq 2)$ vertices has atleast two pendant vertices

3. A Tree with a vertex of degree K has atleast K pendant vertices

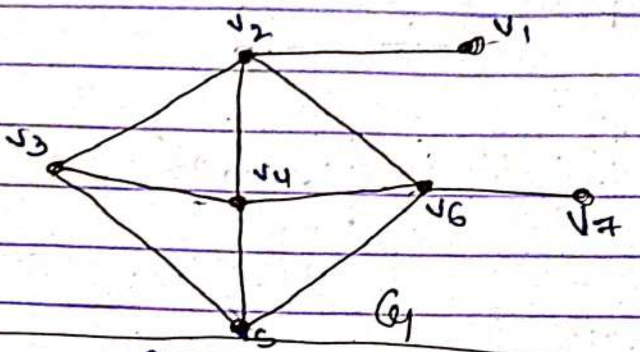
4. Show that there does not exist a self complementary graph with 6 vertices.

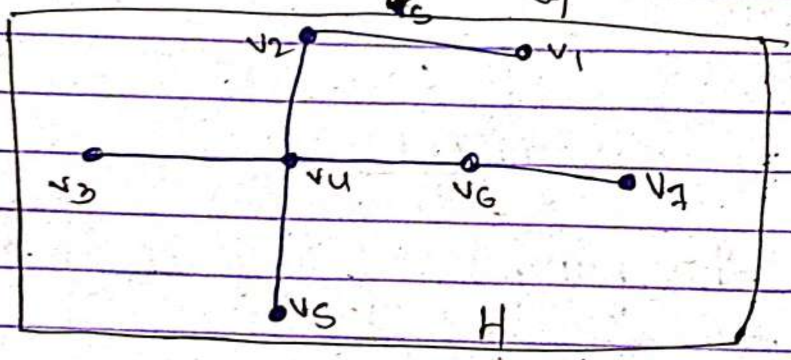
5. Does there exist a simple graph with 6 vertices of the degree 3, 3, 3, 3, 3, 5. If yes then draw such a graph with if no

then appropriate reason

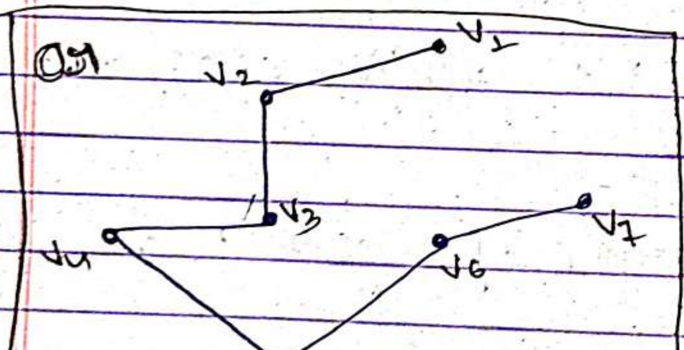
Spanning tree ⇒ Let $G = (V, E)$ be a connected graph. A connected subgraph $H = (V', E')$ is called a spaning tree for G if $V' = V$ and H has no cycles

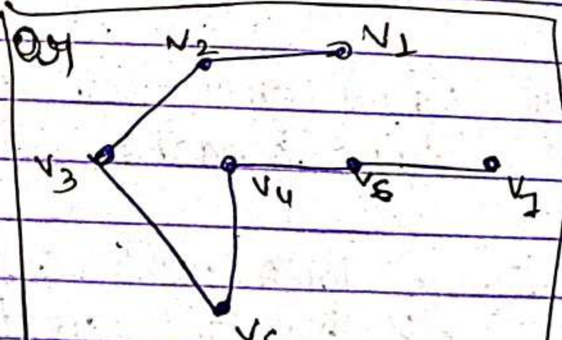
Qus Find spanning tree of the following graph





tree
edges = n-1
no of vertex
No cycle





Date:
Page:

Thearem: Every connected graph G has atleast one spanning tree

Proof If G has no cycle, then G itself a spanning tree of G
If G has cycles, then successively remove one edge from a cycle at a time till the resulting subgraph of G becomes acyclic.

Therefore. H is a spanning tree.
Hence every connected graph G has atleast one spanning tree

Definition :- suppose G be a connected graph then diametre G i.e;

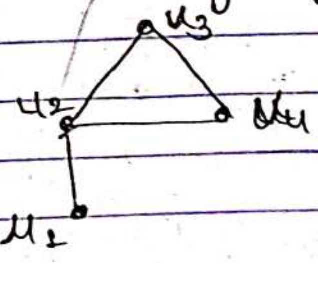
$$\boxed{dia(G) = \max_{u,v \in V(G)} d(u,v)}$$

let $u \in V(G)$, then eccentricity of u i.e;

eccentricity $\boxed{E(u) = \max_{v \in V(G)} d(u,v)}$

Radius of G = $Rad(G) = \min_{u \in V(G)} E(u)$

Qus Find the diameter of, radius and eccentrictity of the following graph



$d(u_1, u_2) = 1$
$d(u_1, u_3) = 2$
$d(u_1, u_4) = 2$
$d(u_2, u_3) = 1$
$d(u_2, u_4) = 1$
$d(u_3, u_4) = 1$

$$\text{dia}(G) = \max_{u,v \in G} d(u,v)$$

$$\boxed{\text{dia}(G) = 2}$$

eccentricity $E(u_1) = 2$
$E(u_2) = 1$
$E(u_3) = 2$
$E(u_4) = 2$

radius of G $\boxed{R(G) = \min E(u)}$ $= 1$

Ques (1) find the ~~d of~~ radius ~~of~~ diameter and eccentricity of the peterson graph.

(2) Find all spanning tree of a graph