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let $e \in G - T$

put $G' = T \cup \{e\}$

then, no. of edges in $G' = |E(T)| + 1$

and G' has exactly 1 cycle, Hence

Number of region w.r.to $G' = 2$

further, $|V(G')| = |V(T)| = |V(G)|$

Hence, Number of vertices

$|V(G')| - |E(G')| + R(G')$

$= n - n + 2$

$= 2$

then in each succesive addition of the edge $e \in G - G'$, no. of edges & no. of region both increas by 1 and Number of vertices does not increases thus eventually leads to G

for G

$$|V| - |E| + R = 2$$

Hence proved

<u>Theorem</u> If G is a connected simple planer graph with n vertices and m edges then $m \leq 3n - 6$

<u>Proof</u> If G is a simple planer graph with n vertices and m edges and d faces then, by euler theorm we have $n - m + d = 2$

since, G is simple <u>degree of each ~~place~~ face is ≥ 3</u>

since, each edge in G is part of atmost 2 faces. Hence $2m \geq 3d$

$$\Rightarrow d \leq \frac{2m}{3}$$

$$2 - n + m \leq \frac{2}{3}m$$

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$$2 \le n - m + \frac{2m}{3}$$

$$2 \le \frac{3n - 3m + 2m}{3}$$

$$6 \le 3n - m$$

$$\boxed{m \le 3n - 6}$$ Hence proved

<u>related example</u> ① K_5 ② $K_{3,3}$

① $n = 5$ vertices, $m = 10$ edges

$$m \le 3n - 6$$

$$10 \le 3 \times 5 - 6$$

$$10 \nleq 9$$

② $n = 6$, $m = 9$

$$m \le 3n - 6$$

$$9 \le 3 \times 6 - 6$$

$$9 \le 18 - 6$$

$$9 \le 12$$

Theorem If G is a connected planer graph with n vertices and m edges such that each face has degree ≥ 4 then $m \le 2n - 4$

<u>Proof</u>

$$2m \geq 4d$$

$$d \le \frac{1}{2}m$$

$$2 - n + m \le \frac{1}{2}m$$

$$2 \le \frac{1}{2}m + n - m$$

$$2 \le \frac{m + 2n - 2m}{2}$$

$$4 \le 2n - m$$

$$m \le 2n - 4$$ proved

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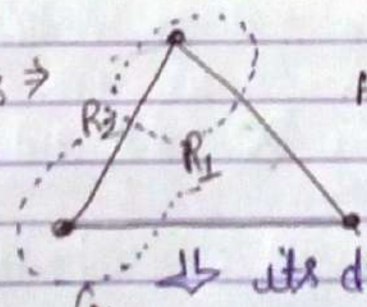
Kuratowski's Theorem :- a graph G is non planer iff it has a sub graph which is homeomorphic to K_5 or $K_{3,3}$.

Homeomorphic Graph :- Two graphs are called homeomorphic if (by adding vertices of degree 2 or contracting) they become isomorphic after adding vertices of degree 2 or by series of contraction or both).

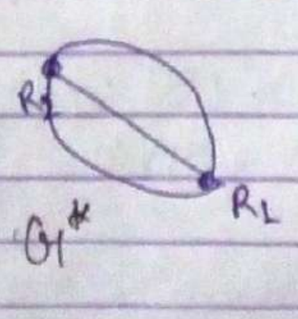
Dual of a planer graph :- let G be a simple planer graph and has r regions i.e; $R_1, R_2, --- R_r$. then $G^* = (V', E')$ where $V' = \{R_1, R_2, --- R_r\}$ for each $e \in G$ $\exists$ an edge $e' \in G^*$ whose endvertices are the region whose boundary contains the edge e. then we say G^* is the dual planer graph of G

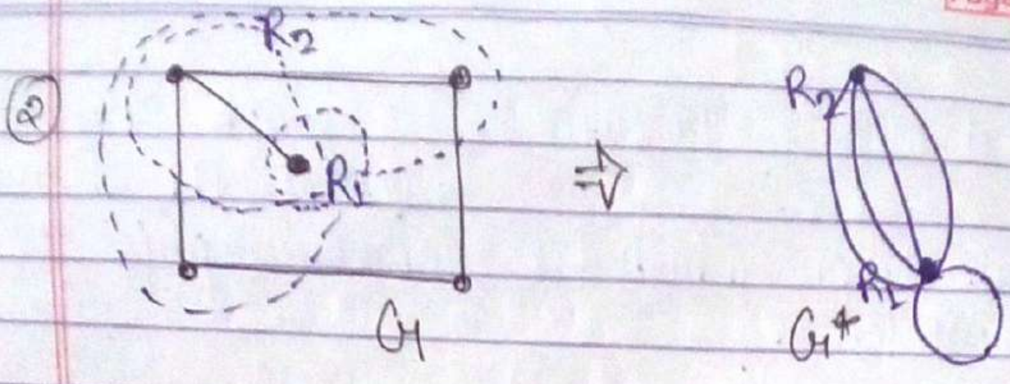
Example $K_3 \Rightarrow$



Here, two region 1 internal and 1 external

$\Downarrow$ its dual graph





Note: If G is planer iff G^* is planer

Dijkstra Algorithm for finding shortest path from a given vertex

Step I: let G be a connected weighted graph and let $u \in V(G)$
for any two vertices x and y in G ($x \neq y$) define temporary weight of $xy = w(x,y)$ if xy is an edge in G

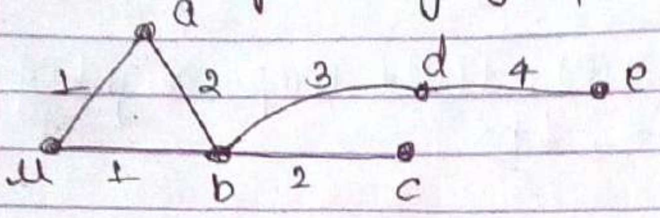
$$xy = \begin{cases} w(x,y) & \text{if } xy \text{ is an edge in } G \\ \infty & \text{if } xy \text{ is not an edge in } G \end{cases}$$

Step I: maintain the set of vertices to which the shortest path from u is known and denote this set by S till we get

$S = V(G)$, $t(u) = 0$ & $t(z)$ is the temporary weight of uz if $u \neq z$

Step II. let $t(v) = \min t(z)$ and $u \neq z$
$S = \{u, v\}$ $\quad S = V(G)$

Example Find the shortest path from the vertex u in the following graph



selfloop = 0

	u	a	b	c	d	e
u	0	1	1	∞	∞	∞
a	1	0	2	∞	∞	∞
b	1	2	0	2	3	∞
c	∞	∞	2	0	∞	∞
d	∞	∞	3	∞	0	4
e	∞	∞	∞	∞	4	0

Let $S = \{u\}$

$t(u) = 0 \quad t(a) = 1 \quad t(b) = 1 \quad t(c) = \infty \quad t(d) = \infty$
$t(e) = \infty$

$$\min_{z \neq u}\{t(z)\} = 1 = t(a) = t(b)$$

Now, $S = \{u, a\}$

$$t(b) = \min\{t(b), t(a) + \text{temp. wt. of } ab\}$$
$$= \min\{1, 1+2\}$$
$$= \min\{1, 3\}$$
$$= 1$$

$$t(c) = \min\{t(c), t(a) + \text{temp weight of } ca\}$$
$$= \min\{\infty, 1+\infty\}$$
$$= \infty$$

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$l(d) = \min\{l(d), l(a) + \text{temp. weight of } ad\}$
$= \min\{\infty, 1+\infty\}$
$= \infty$

$l(e) = \min\{l(e), l(a) + \text{temp. weight of } ea\}$
$= \min\{\infty, 1+\infty\}$
$= \infty$

Hence, $S = \{u, a, b\}$.

Now,

$l(c) = \min\{l(c), l(b) + \text{temp. wt of } cb\}, l(a) + l(ac)$
$= \min\{\infty, 1+2, 1+\infty\}$
$= \min\{\infty, 3, \infty\}$
$= 3$

$l(d) = \min\{l(d), l(b) + l(bd), l(a) + l(ad)\}$
$\min\{\infty, 1+3, 1+\infty\}$
$= 4$

$l(e) = \infty$

$\min\{l(c), l(d), l(e)\}$

Hence, $S = \{u, a, b, c\}$

$l(d) = \min\{l(d), l(c) + l(cd), l(b) + l(bd), l(a) + l(ad)\}$
$= \min\{\infty, \infty+\infty, 1+3, 1+\infty\}$
$= 4$

$l(e) = \infty$

$S = \{u, a, b, c, d\}$

$l(e) = 8 = d(u, e)$

$S = \{u, a, b, c, d, e\}$

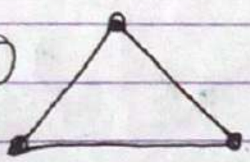
Z - all variable
u, a, b, c, d, e

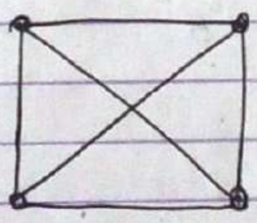
	$d(u, z)$
u	0
a	1
b	1
c	3
d	4
e	8 = Ans

tt(ac)}

Proper Colouring :- suppose G is a graph, a colouring of the vertices of G in such way that no adjacent vertices get the same colour, is called a proper colouring of the graph.

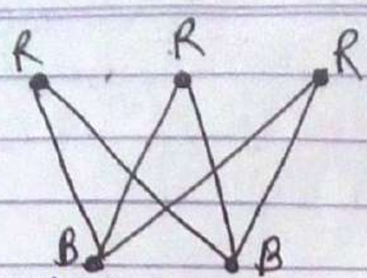
Chromatic Number :- Minimum number of colours required for a proper colouring of the graph, is called chromatic number. It is denoted by - $\chi(G)$

Example ①  $\chi(G) = 3$

② K_4  $\chi(G) = 4$

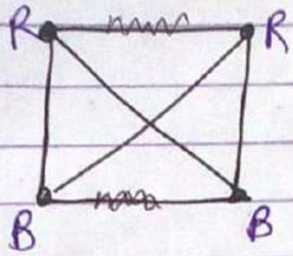
general $\boxed{\chi(K_n) = n}$

③ $K_{3,2}$



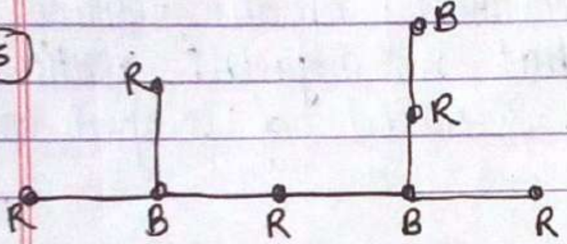
$\chi(G) = 2$

④ $K_{2,2}$ =



$\chi(G) = 2$

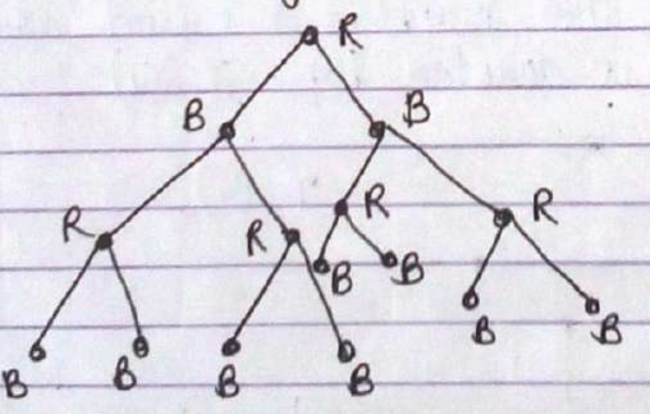
⑤



$\chi(G) = 2$

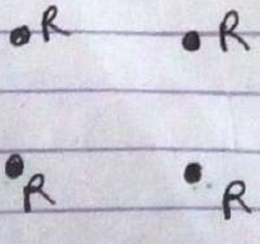
Note: for any tree $\chi(G) = 2$

⑥



$\chi(G) = 2$

⑦



$\chi(G) = 1$

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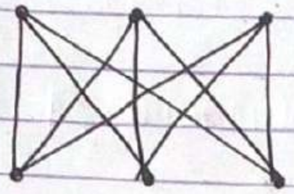
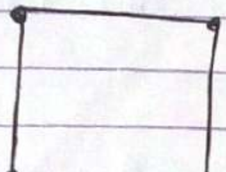
8. If G is a cyclic graph of odd length is $\chi(G)$ 2 or less than 2

for example : $K_{3,3}$



9.



$\chi(G) = 2$

Chromatic Polynomials :- let G be a graph and $P_G(\lambda)$ is the number of proper colors of G in λ or less colors, then $P_G(\lambda)$ is a polynomial in λ called chromatic polynomial

general complete graph with n vertices formula

① $\boxed{P_{K_n}(\lambda) = \lambda(\lambda-1) --- (\lambda-n+1)}$

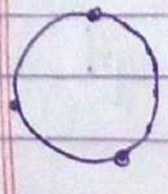
$P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)$

② If T is a tree with n vertices, then

$\boxed{P_T(\lambda) = \lambda(\lambda-1)^{n-1}}$

③ If G is a cycle of length n then

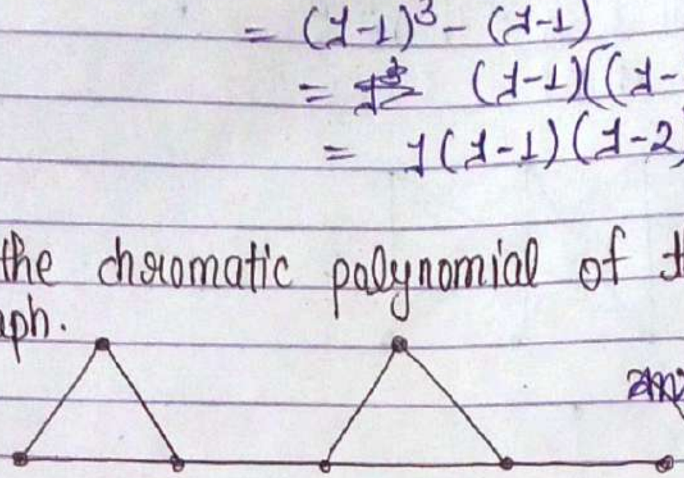
$\boxed{P_G(\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)}$



$$P_3(\lambda) = (\lambda-1)^3 + (-1)^3(\lambda-1)$$
$$= (\lambda-1)^3 - (\lambda-1)$$
$$= \cancel{\lambda^3}\ (\lambda-1)[(\lambda-1)^2-1]$$
$$= \lambda(\lambda-1)(\lambda-2)$$

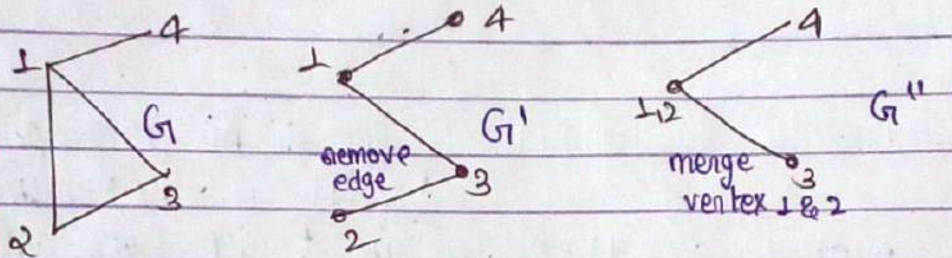
Ques Find the chromatic polynomial of the following graph.

(1)



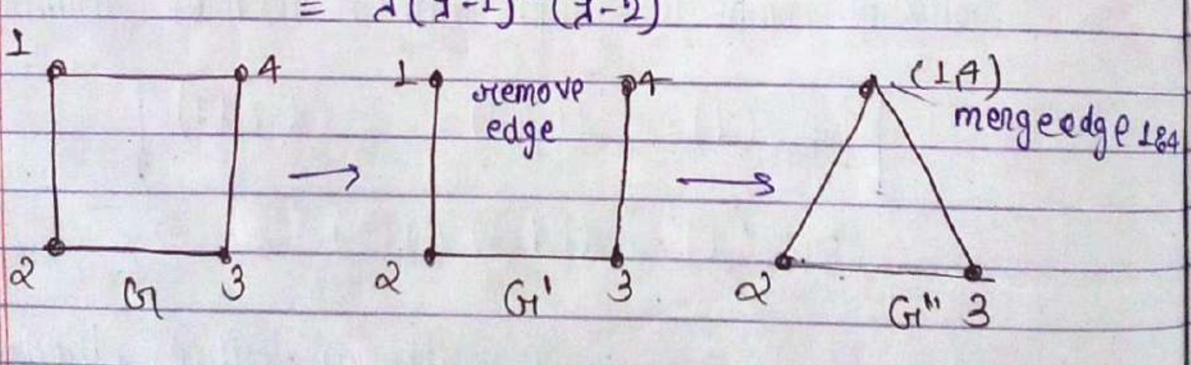
~~Ans: $\lambda(\lambda-1)^2\{2(\lambda-1)+(\lambda-2)(\lambda-3)-1\}$~~

(2)



$$P_G(\lambda) = P_{G'}(\lambda) - P_{G''}(\lambda)$$
$$= \lambda(\lambda-1)^3 - \lambda(\lambda-1)^2$$
$$= \lambda(\lambda-1)^2(\lambda-2)$$

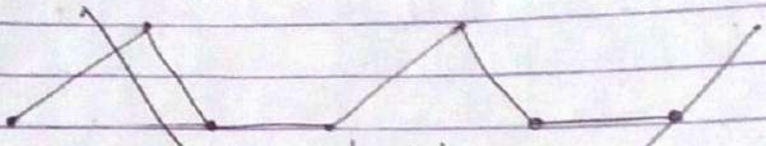
(3)



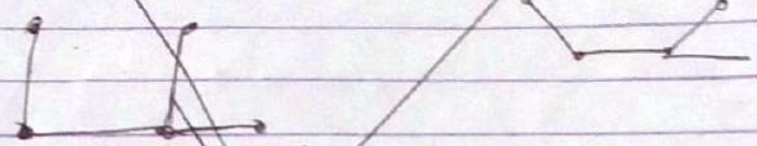
$$P_G(\lambda) = P_{G'}(\lambda) - P_{G''}(\lambda)$$
$$= \lambda(\lambda-1)^3 - [(\lambda-1)^3 + (-1)^3(\lambda-1)]$$
$$= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)[(\lambda-1)^2 - (\lambda-2)]$$
$$= \lambda(\lambda-1)[\lambda^2+1-2\lambda-\lambda+2]$$

$= t(t-1)\,[t^2 + 3 - 3t]$ ans

$\cancel{= t(t-1)\,[t^2 - 2t}$



$pG'(t)$



del^n

$pG''(t)$

$pG(t) = pG'(t) - pG''(t)$

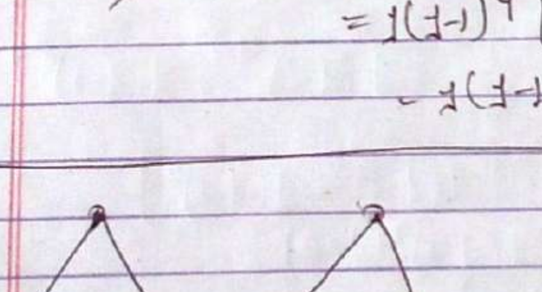
$= t(t-1)^{7-1} - t(t-1)^{5-1}$

$= t(t-1)^6 - t(t-1)^4$

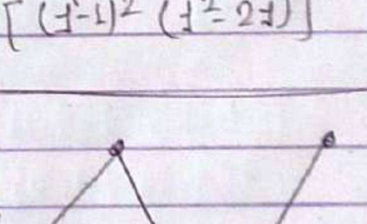
$= t(t-1)^4\,[(t-1)^2 - 1]$

$= t(t-1)^4\,[t^2 + 1 - 2t - 1]$

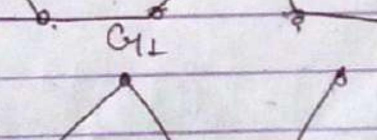
$= t(t-1)^2\,[(t-1)^2\,(t^2 - 2t)]$



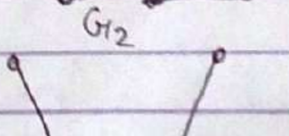
G_1



G_2



G_3



G_4

$pG(t) = [pG_1(t) - pG_2(t)] - [pG_3(t) - pG_4(t)]$

$[t(t-1)^{7-1} - t(t-1)^{6-1}] - [t(t-1)^{6-1} - t(t-1)^{5-1}]$

No chromatic polynomial of isolated graph

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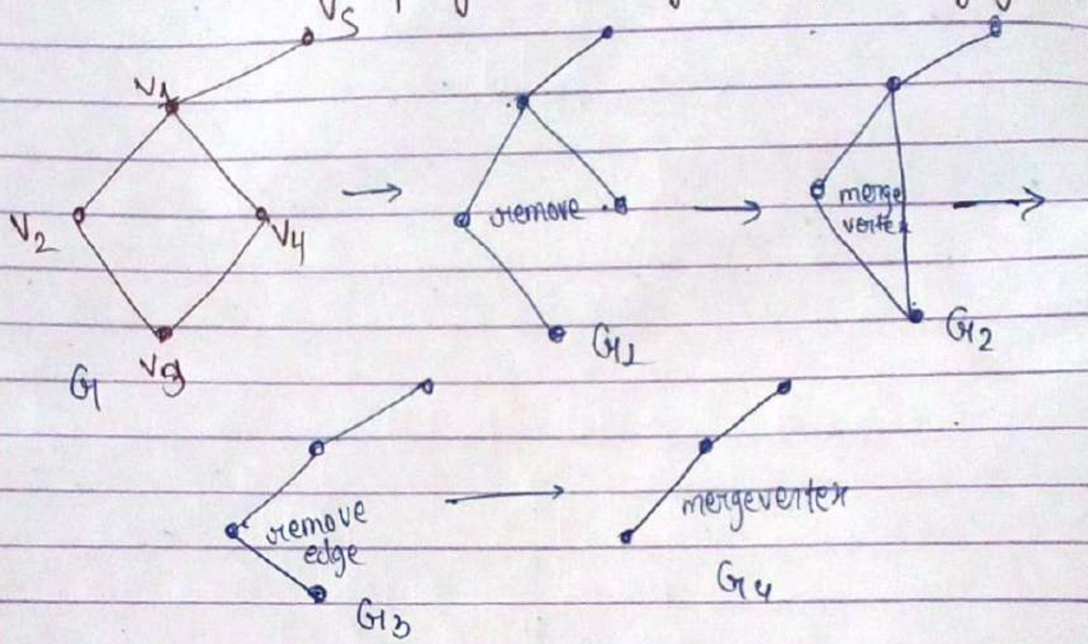
$$= \left[\lambda(\lambda-1)^6 - \lambda(\lambda-1)^5\right] - \left[\lambda(\lambda-1)^5 - \lambda(\lambda-1)^4\right]$$
$$= \lambda(\lambda-1)^5\left[\lambda-1-1\right] - \lambda(\lambda-1)^4\left[\lambda-1-1\right]$$
$$= \lambda(\lambda-1)^5(\lambda-2) - \lambda(\lambda-1)^4(\lambda-2)$$
$$= \lambda(\lambda-2)(\lambda-1)^4\left[\lambda-1-1\right] \Rightarrow \lambda(\lambda-2)^2(\lambda-1)^4 \text{ Ans}$$

Que Find the chromatic polynomial of the fallowing graph.



$$p_G(\lambda) = \left[pG_1(\lambda) - pG_2(\lambda)\right] - \left[pG_3(\lambda) - pG_4(\lambda)\right]$$
$$= \left[\lambda(\lambda-1)^{5-1} - \lambda(\lambda-1)^{4-1}\right] - \left[\lambda(\lambda-1)^{4-1} - \lambda(\lambda-1)^{3-1}\right]$$
$$= \left[\lambda(\lambda-1)^4 - \lambda(\lambda-1)^3\right] - \left[\lambda(\lambda-1)^3 - \lambda(\lambda-1)^2\right]$$
$$= \lambda(\lambda-1)^3(\lambda-2) - \lambda(\lambda-1)^2(\lambda-2)$$
$$= \lambda(\lambda-2)^2(\lambda-1)^2$$

$$= \left[\lambda(\lambda-1)^{5-1} - (\lambda-1)^4 + (-1)^4(\lambda-1)\right] - \left[\lambda(\lambda-1)^{4-1} - \lambda(\lambda-1)^{3-1}\right]$$
$$= \left[\lambda(\lambda-1)^4 - (\lambda-1)^4 - (\lambda-1)\right] - \left[\lambda(\lambda-1)^3 - \lambda(\lambda-1)^2\right]$$
$$= (\lambda\cancel{\lambda}-1)\left[\lambda(\lambda-1)^3 - (\lambda-1)^3 - 1\right] - \lambda(\lambda-1)^2(\lambda-2)$$
$$= (\lambda-1)\left[\lambda(\lambda-1)^3 - (\lambda-1)^3 - 1 - \lambda(\lambda-1)(\lambda-2)\right]$$
$$= (\lambda-1)\left[(\lambda-1)\left\{\lambda(\lambda-1)^2 - (\lambda-1)^2 - \lambda(\lambda-2)\right\} - 1\right]$$

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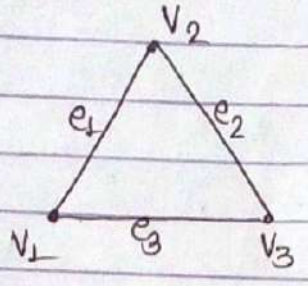
Matching :- let $G = (V,E)$ be a graph a subset M of E is called a matching if no edges in M have the same end vertices.

Example



set of edges = $\{e_1, e_2, e_3\}$

$M_1 = \{e_1\}$ $M_2 = \{e_2\}$ $M_3 = \{e_3\}$ is matching but the set $\{e_1, e_2\}$ in G is not matching

OR

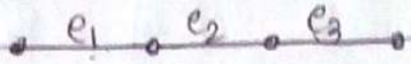
A matching in a graph is a subset of edges in which no two edges are adjacent.

Maximal matching :- a maximal matching M in $G = (V,E)$ is called maximal if M is not properly contain in any other matching in G

Complete matching :- a matching M is called complete matching if every vertex in G is incident on all edges in M.

Matching Number :- Matching NO. = NO. of edges in a largest maximal matching

Ex ⇒



$M_1 = \{e_2\}$ is matching but not maximal matching

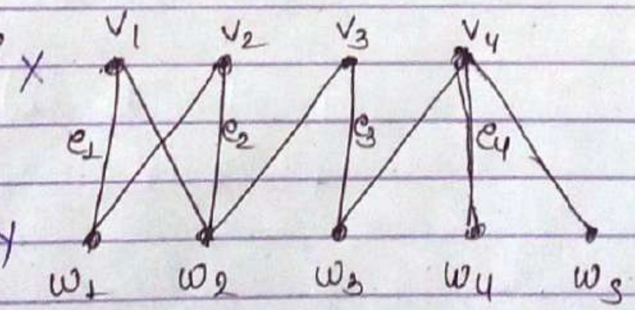
$M_2 = \{e_1, e_3\}$ is maching and maximal (largest) matching

Hence, matching number of G is 2.
It is also complete matching.

Note: If G has a complete matching M then M is also a largest maximal matching but conversely need not be true.

Perfect matching :- In a bipartite graph (X, Y, E) a complete matching from X to Y is a matching M such that every vertex in X is incident to an edge in M and a perfect matching is a matching that is complete from X to Y as well as from Y to X has the same number of element

Example :

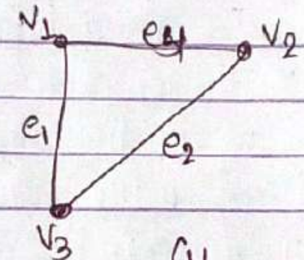


$M = \{e_1, e_2, e_3, e_4\}$ is matching and perfect matching

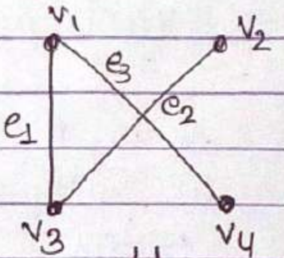
Augmenting path :- let $G = (V_1 \cup V_2, E)$ be a bipartite graph and M be a matching from V_1 to V_2. A path is called augmenting path w.r.to a matching M if the edges in path alternatively belong to E-M and M, starting and terminating with an edge in E-M

(Δ) Symmetric diffrence of graph :- let G and H be any two grap[h]
then, $G \Delta H = (V, E)$ is also a graph where
$E = (E_1 - E_2) \cup (E_2 - E_1)$

Example :



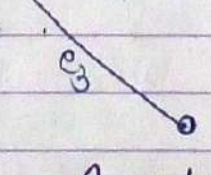
G1



H

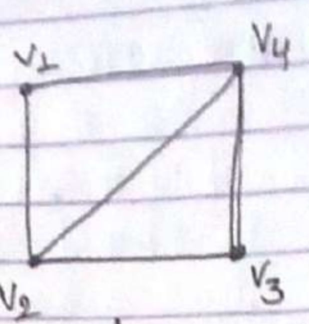
$G \Delta H = (E_1 - E_2) = e_4$
$(E_2 - E_1) = e_3$

$= (E_1 - E_2) \cup (E_2 - E_1)$

$=$ 

Vertex covering :- A subset B of vertices in a graph $G = (V, E)$ is called a vertex covering of G if every edge in ~~vector~~ G is incident to a vertex in B

Example :-



$V = \{v_1, v_2, v_3, v_4\}$ subset $B = \{v_2, v_4\}$

Hence B is vertex covering of G
$B_1 = \{v_1, v_2, v_3\}$ is also vertex covering in G

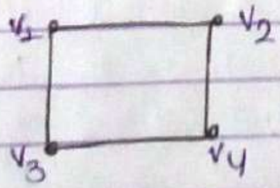
Remark :- Number of vertices in any vertex covering of G ≥ Number of edges in any matching

Min-Max Theorem

Statement : Number of edges in the largest possible matching in equal to the number of vertices in the smallest vertex covering of G

Definition ⇒ let G (V,E) be a graph A subset $A \subseteq V$ is called independent in G If no two vertices are end point of same edges in G

Example :

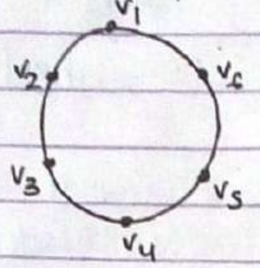


maximal independent set $A = \{v_1, v_3\}$ is independent set of G

$A = \{v_1, v_2, v_3\}$ is not independent set.

Maximal independent set : an independent set in G is a maximal independent set if it is not properly contained in any other independent set

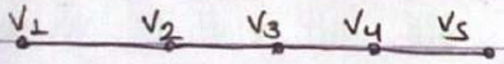
Example :



$A = \{v_1, v_3\}$ is independent set. not maximal also ~~but~~ $\{v_1, v_3, v_5\}$ is max. independent set.

Independence Number : Independence number $\alpha(G)$ of a graph is the cardinality of largest maximal independent set

Example : $A = \{v_1, v_3, v_5\}$ is max. independent set
Hence, $\alpha(G) = 3$



$A = \{v_1, v_3, v_5\}$ is maximal independent set.
Hence, $\alpha(G) = 3$

Saturated vertex :- If M is a matching in $G(V,E)$ a vertex $v \in V$ is called a saturated vertex w.r. to M If $\exists$ $e \in M$ such that one end vertex of e in v.

(capacity ≥ flow)

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Network :- a Network N is a directed graph with two distinguish vertex s and t are called source (s) and sink respectively such that for each directed edge $e = (v_i, v_j)$ there is a unique number C_{ij} called capacity of the edge associated to it and denoted by $N = (G^+, C)$

$$C : E \longrightarrow \mathbb{R}^+$$

$$C(e) = +\text{ive real number} = C_{ij}$$

Flow :- A flow f along a network associated with a unique number f_{ij} to each directed edge $e = (v_i, v_j)$ such that

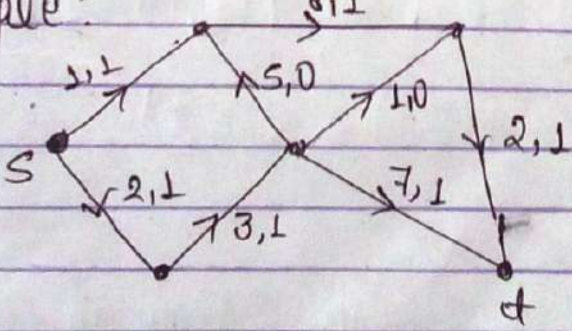
(i) $f_{ij} \le C_{ij}$ for each edge

(ii) for each vertex $v \neq s, t$

total flow up to $v_K = \sum_{(v_i, v_K) \text{ in an edge}} f_{iK} = \sum_{(v_K, v_j) \text{ in an edge}} f_{Kj}$

then total flow value from s to t $= \sum_{e = (u,t) \text{ in an directed edge}} f(u, t)$

For example :



capacity ≥ flow

If the value of flow and sink are same. then we can say that flow is feasible

Capacitated network :- A diagraph with N integer value function c defined on its set of edges (arc) is called a capacited network

Cuts in a capacitated network : consider any partition of the vertex set of the capacitated network $G = (V, E)$ in to two sets S and T such that the source is in S and sink is in T.

→ The set $(S,T) = \{(i,j) \in E : i \in S, j \in T\}$ is called a cut in the network.

→ since No flow can be send from the source to sink if all the edges (arc) in cut are deleted

→ The sum of the capacity of all the arc in cut (s,t) is the capacity of $c(S,T)$ of the cut.

→ If cut is called a minimum cut if its capaicty does not exceed the capacity of any other cut.

→ If f is a feasible flow in the network. then sum of the flows along all the arcs in cut (S,T) is the flow $f(S,T)$ along the cut

$$0 \leq f(S,T) \leq c(S,T)$$

Thearem : If f is any feasible flow in a capacitated network G and if (S,T) is any cut in the network

then $f(G) = f(S,T) - f(T,S)$

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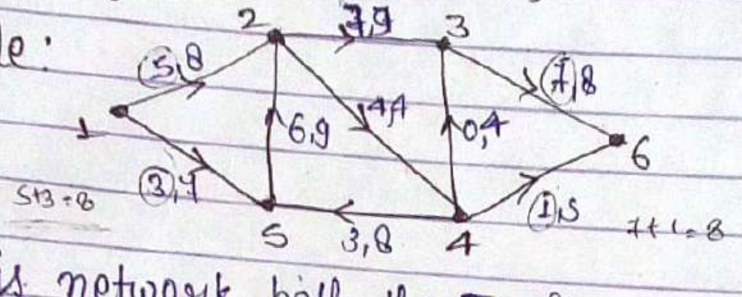
Carallary: If f is any feasible flow and if (S,T) is any cut $f(G) \leq C(S,T)$

Example:



In this network both the In flow in to the sink and out flow from the source for the current flow f are 8. which is the flow value.

① If $S = \{1,2,3\}$ & $T\{4,5,6\}$ the flow value ~~is also equal to~~

flow $f(G) = f(S,T) - f(T,S)$

$= (3+4+7) - (0+6)$

$= 8$

Capacity $C(S,T) = 7+4+8 = 19$

$f(G) = 8$

$f(G) \leq C(S,T)$

$8 \leq 19$

② If $S = \{1,4,5\}$, $T = \{2,3,6\}$ then the flow

$f(G) = f(S,T) - f(T,S)$

$\cancel{= (5+0+6) - (0+0+0)}$

$= (5+0+0+0+0+1+6+0+0) - (0+4+0+0+0+0+0+0+0$

$= 12-4$

$= 8$