→ Pitch circle

Two degree of freedom in spring - mass system

To find -

a) ~~Natural~~ fre DEOM

b) Natural frequency

c) ~~Node~~ Mode shape

$\mathcal{L} = T - V$

$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

$V = \frac{1}{2} k (x_2 - x_1)^2$

$\Rightarrow L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1)^2$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$

$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2} \times 2 \times m \times \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m \cdot \ddot{x}_1$

$\frac{\partial L}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-\frac{1}{2} k \left[x_1^2 + x_2^2 - 2 x_1 x_2\right]\right)$

$= -k (x_1 - x_2)$

$\Rightarrow m \ddot{x}_1 + k x_1 - k x_2 = 0$ —— (1)

Now,

$\frac{\partial L}{\partial \dot{x}_2} = m \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m \ddot{x}_2$

$\frac{\partial L}{\partial x_2} = \frac{\partial}{\partial x_2}\left(-\frac{1}{2} k \left(x_1^2 + x_2^2 - 2 x_1 x_2\right)\right)$

$$\frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$$

$$\Rightarrow m\ddot{x}_2 + kx_2 - kx_1 = 0 \quad \text{———— (ii)}$$

Eqn (i) & (ii) —

$$m\ddot{x}_1 + 0\ddot{x}_2 + kx_1 - kx_2 = 0$$

$$0\ddot{x}_1 + m\ddot{x}_2 - kx_1 + kx_2 = 0$$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{\text{mass matrix}} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\text{stiffness matrix}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$[M]_{2\times2}\{\ddot{X}\}_{2\times1} + [k]_{2\times2}\{x\}_{2\times1} = \{0\}_{2\times1}$$

$$\ddot{X} = -\omega^2 \cdot X$$

$$\Rightarrow -\omega^2[M]\{x\} + [k]\{x\} = 0$$

$$([k] - \omega^2[M])\{x\} = 0$$

$$\det([k] - \omega^2[M]) = 0$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} + \begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix} = 0$$

$$\begin{vmatrix} k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$$

$$(k - m\omega^2)^2 - k^2 = 0.$$

$$\cancel{K^2} + m^2\omega^4 - 2Km\omega^2 - \cancel{K^2} = 0$$

$$m^2\omega^4 - 2Km\omega^2 = 0$$

$$\cancel{m^2\omega^{4\,2} = 2Km\omega^2}$$

$$\cancel{m\omega^2 = 2K\omega}$$

$$\omega^2\left(m\omega^2 - 2Km\right) = 0$$

$$\boxed{\omega = 0}$$

$$m\omega^2 - 2Km = 0$$

$$\boxed{\omega = \sqrt{\frac{2K}{m}}}$$

Calculation of frequency by matrix method.

$$\Rightarrow m\ddot{x}_1 + Kx_1 - Kx_2 = 0$$

$$m\ddot{x}_2 - Kx_1 + Kx_2 = 0$$

$$x_1 = A_1 \sin(\omega t + \phi)$$

$$x_2 = A_2 \sin(\omega t + \phi)$$

$$\dot{x}_1 = A_1 \cos(\omega t + \phi)\omega$$

$$\ddot{x}_1 = -A_1\omega^2 \sin(\omega t + \phi)$$

$$\dot{x}_2 = A_2 \cancel{\sin}\cos(\omega t + \phi)\omega$$

$$\ddot{x}_2 = -A_2 \sin(\omega t + \phi)\omega^2$$

$$\cancel{\Rightarrow m\omega^2 +}$$

$$\Rightarrow \left[-m\omega^2 A_1 + KA_1 - KA_2\right]\left(\sin \omega t + \phi\right) = 0$$

$$\Rightarrow \left[-m\omega^2 A_2 - KA_1 + KA_2\right]\left(\sin \omega t + \phi\right) = 0$$

$$\frac{A_1}{A_2} = \frac{K}{K - m\omega^2} \quad \text{—— (iii)}$$

$$\left(K - m\omega^2\right) A_1 = KA_2$$

Similarly,

$$\frac{K - m\omega^2}{K} = \frac{A_1}{A_2} \quad \text{—— (iv)}$$

Equating eqⁿ (III) & (IV) -

$$\frac{K - m\omega^2}{K} = \frac{K}{K - m\omega^2}$$

$$K^2 - (K - m\omega^2)^2 = 0$$

$$\boxed{\begin{aligned} \omega^2 &= 0 \\ \omega &= \sqrt{\frac{2K}{m}} \end{aligned}}$$

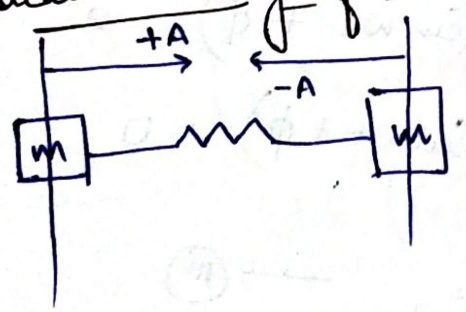
$\omega_1 = 0$

$\omega_2 = \sqrt{\frac{2K}{m}}$.

Ist mode shape. $\leftarrow \quad \frac{A_{11}}{A_{21}} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

IInd mode shape $\leftarrow \quad \frac{A_{12}}{A_{22}} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$.

Physical meaning of mode shape :-



IInd mode shape.