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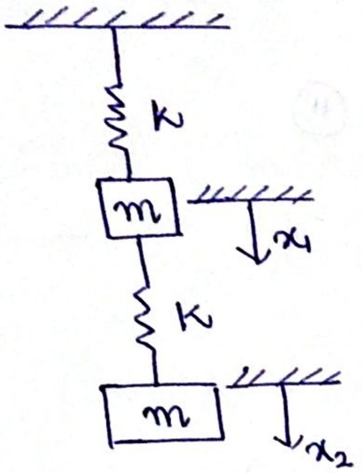
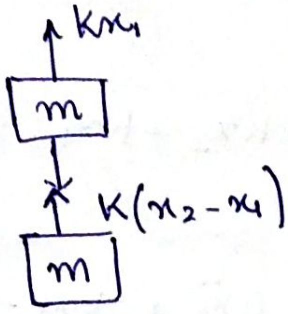
Assignment

Q)

(i)





Assuming ; $x_2 > x_1$

$$L = T - V$$

$$\begin{Bmatrix} q_1 = x_1 \\ q_2 = x_2 \end{Bmatrix} \qquad \because \; T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}Kx_1^2 + \frac{1}{2}K(x_2 - x_1)^2$$

$$\because \; i = 1, 2$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Now, $$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

& $$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \left[\frac{1}{2}Kx_1^2 + \frac{1}{2}K(x_2 - x_1)^2\right]$$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \; ; \; \frac{\partial L}{\partial x_1} = -\left[\frac{1}{2}\cdot 2x_1 + \frac{1}{2}\times 2K(x_2 - x_1)\cdot(0-1)\right]$$

$$\frac{\partial L}{\partial x_1} = -Kx_1 + K(x_2 - x_1)$$

Now;

$$\frac{d}{dt}(m\dot{x}_1) + kx_1 - k(x_2 - x_1) = 0$$

$$\Rightarrow m\ddot{x}_1 + kx_1 - k(x_2 - x_1) = 0 \quad \text{------} (A)$$

$$\Rightarrow m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad \text{------} (I)$$

Similarly,

$$m\ddot{x}_2 + kx_2 - kx_1 = 0 \quad \text{------} (II)$$

So;

$$m\ddot{x}_1 + 0\ddot{x}_2 + 2kx_1 - kx_2 = 0$$

$$0\ddot{x}_1 + m\ddot{x}_2 + kx_2 - kx_1 = 0$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

We know that,

$$\ddot{x} + \omega_n^2 x = 0$$

$$\ddot{x} = -\omega^2 x$$

$$[M]_{2\times2} \{\ddot{x}\}_{2\times1} + [K]_{2\times2} \{x\}_{2\times1} = \{0\}$$

$$\left[[K] - \omega^2[M]\right]\{x\} = 0$$

$$\det(K - \omega^2 M) = 0$$

$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} + \begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} = 0$$

$$(2k - m\omega^2)(k - m\omega^2) - k^2 = 0.$$

(ii) Mode shape:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + kx_2 - kx_1 = 0$$

$$x_1 = A_1 \sin(\omega t + \phi) \quad ; \quad x_2 = A_2 \sin(\omega t + \phi)$$

$$\ddot{x}_1 = -A_1 \omega^2 \sin(\omega t + \phi)$$
$$\ddot{x}_2 = -A_2 \omega^2 \sin(\omega t + \phi)$$

$$\Rightarrow \text{So} ; \; (-m\omega^2 A_1 + 2kA_1 - kA_2)(\sin(\omega t + \phi)) = 0$$

$$-m\omega^2 A_1 + 2kA_1 - kA_2 = 0 \quad \text{—— (III)}$$

$$\Rightarrow (-m\omega^2 A_2 + kA_2 - kA_1)(\sin(\omega t + \phi)) = 0$$

$$-m\omega^2 A_2 + kA_2 - kA_1 = 0 \quad \text{—— (IV)}$$

from eqn (III) –

$$(-m\omega^2 + 2k) A_1 = kA_2$$

$$\frac{A_1}{A_2} = \frac{k}{-m\omega^2 + 2k}$$

from eqn. (IV) –

$$(-m\omega^2 + k) A_2 = kA_1$$

$$\frac{A_1}{A_2} = \frac{k - m\omega^2}{k}$$

So;

$$\frac{A_1}{A_2} = \frac{k}{-m\omega^2 + 2k} = \frac{k - m\omega^2}{k}$$

$$k^2 = 2k^2 - 3km\omega^2 + m^2\omega^4$$

$$k^2 - 3km\omega^2 + m^2\omega^4 = 0$$

let $\omega^2 = \lambda$

$$k^2 - 3km\lambda + m^2\lambda^2 = 0$$

$$m^2\lambda^2 - 3km\lambda + k^2 = 0$$

$$\lambda = \frac{3km \pm \sqrt{(3km)^2 - 4m^2k^2}}{2m^2}$$

$$\lambda = \frac{3}{2}\frac{k}{m} \pm \frac{\sqrt{5}}{2}\cdot\frac{k}{m^2}$$

$$\lambda_1 = \frac{3km + \sqrt{5}\,km}{2m^2} \quad \& \quad \lambda_2 = \frac{3km - \sqrt{5}\,km}{2m^2}$$

$$\lambda_1 = 2.6\frac{k}{m} \quad \& \quad \lambda_2 = 0.38\frac{k}{m}$$

$$\boxed{\omega_1 = \sqrt{\frac{2.6k}{m}} = 1.61\sqrt{\frac{k}{m}}}$$

$$\boxed{\omega_2 = 0.62\sqrt{\frac{k}{m}}}$$

Now; for mode shapes -

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \frac{k}{2k - \cancel{m} \times 2.6 \times \frac{k}{\cancel{m}}} = \frac{k}{2k - 2.6k}$$

first mode shape -

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix} = \frac{-10}{6} = \begin{Bmatrix} -5 \\ 3 \end{Bmatrix} \text{ or } \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix}$$

Similarly,

$$\frac{A_{12}}{A_{22}} = \frac{k}{2k - \cancel{m} \times (0.38) \times \frac{k}{\cancel{m}}} = \frac{100}{162}$$

Second mode shape -

$$\frac{A_{12}}{A_{22}} = \begin{Bmatrix} 50 \\ 81 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}$$

(iii) Modal matrix $[P] = \left[\{x_1\} \quad \{x_2\}\right]$

$$= \begin{bmatrix} -1 & 1 \\ 0.6 & 1.62 \end{bmatrix}$$

Decoupling of eq^n –

$m\ddot{x}_1 + 0\ddot{x}_2 + 2kx_1 - kx_2 = 0$ —— (I)

$m\ddot{x}_2 + 0\ddot{x}_1 + kx_2 - kx_1 = 0$ —— (II)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}.$$

$M_{11}\ddot{y}_1 + k_{11}y_1 = 0$

$M_{22}\ddot{y}_2 + k_{22}y_2 = 0$.

$\Rightarrow \quad M\ddot{X} + kX = 0$

Put $X = PY$

$$\underset{2\times2}{[M]}\underset{2\times1}{[P]}\underset{2\times1}{\ddot{Y}} + \underset{2\times2}{[k]}\underset{2\times1}{[P]}\underset{2\times1}{Y} = 0$$

Now, multiplying by P^T;

$P^T[M][P]\ddot{Y} + P^T[k][P]Y = 0$ (Decouple for DEOM)

$\because \; P^T[M][P]\ddot{Y} = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$

$M_{11} = X_1^T M X_1 = \{-1 \; 0.6\} \begin{Bmatrix} m & 0 \\ 0 & m \end{Bmatrix} \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}$

$M_{11} = 0.28M$

$M_{22} = X_2^T M X_2 = \{1 \; 1.62\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix}$

$M_{22} = 0.2\,M$

& for;

$$K_{11} = X_1^T K X_1$$

$$K_{22} = X_2^T K X_2$$

$$\Rightarrow P^T = [K][P]Y = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

$$K_{11} = \{-1 \quad 0.6\} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 1.62 \end{Bmatrix}$$

$$K_{11} = 0.008\, k$$

$$K_{22} = \{1 \quad 1.62\} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix}$$

$$K_{22} = -0.1 k$$

(iv) Orthogonality -

$$X_1 = \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix} \quad \& \quad X_1^T = \{-1 \quad 0.6\}$$

$$\Rightarrow \{-1 \quad 0.6\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 1.63 \end{bmatrix} = 0$$

$$-m + 0.99m = 0 \quad \Rightarrow \quad -m + m = 0$$

Now;

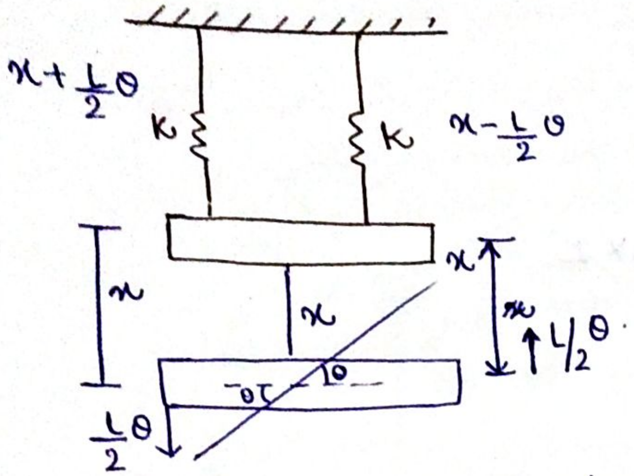
$$\{1 \quad 1.62\} \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix} = 0$$

$$\{0.38\, k \quad 0.62k\} \begin{Bmatrix} -1 \\ 0.6 \end{Bmatrix} = 0$$

$$-0.38\, k + 0.376\, k = 0$$

Verified.

Hence, orthogonal.



$$U_1 = \frac{1}{2}k\left(x + \frac{L}{2}\theta\right)^2$$

$$U_2 = \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2$$

$$U_1 + U_2 = \frac{1}{2}k\left(x^2 + \frac{L^2}{4}\theta^2 + x^2 + \frac{L^2\theta^2}{4}\right)$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{cm}\omega^2$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} - \cancel{\frac{\partial V}{\partial \dot{x}}}^{\,0}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \cancel{\frac{\partial L}{\partial x}} \;\cancel{=0}\; m\ddot{x}$$

$\cancel{m\ddot{x}}$

$$\frac{\partial T}{\partial x} = 0 \quad , \quad \frac{\partial V}{\partial x} = -2Kx$$

$$\Rightarrow \frac{\partial L}{\partial x} = \cancel{\frac{\partial T}{\partial x}}^{\,0} - \frac{\partial V}{\partial x}$$

$$m\ddot{x} = -2Kx$$

$\cancel{m\ddot{x}}$ -

$$m\ddot{x} + 2Kx = 0 \quad \text{———} \quad (1)$$

$$\frac{\partial L}{\partial \dot\theta} = \frac{\partial T}{\partial \dot\theta} - \frac{\partial V}{\partial \theta}$$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_{cm} \dot\theta^2$$

$$\frac{\partial T}{\partial \theta} = 0 + \frac{1}{\cancel{2}} I_{cm} \dot\theta \times \cancel{2}$$

$$\frac{\partial T}{\partial \theta} = I_{cm} \cdot \dot\theta$$

$$\Rightarrow \frac{\partial V}{\partial \theta} = \frac{1}{2} K L^2 \theta$$

~~A~~ finally;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$I_{cm} \cdot \alpha + \frac{1}{2} K L^2 \theta = 0 \quad \text{———} \quad (11)$$

∵ Both the eqn's have their individual variables therefore they are already decoupled.

$$\omega_1^2 = \frac{2k}{m}$$

$$\omega_2^2 = \frac{6k}{m}$$