Reynolds Transport Theorem (RTT)

As we know that the laws of Physics are basically applicable for a particle or a Control mass system or say closed system. e.g. mass, momentum and energy equations.

(CMS) Control Mass System

Conservation of mass, momentum and energy equations

(CVS) Control volume System

Eulerian approach requires a mathematical modeling in differential and integral form

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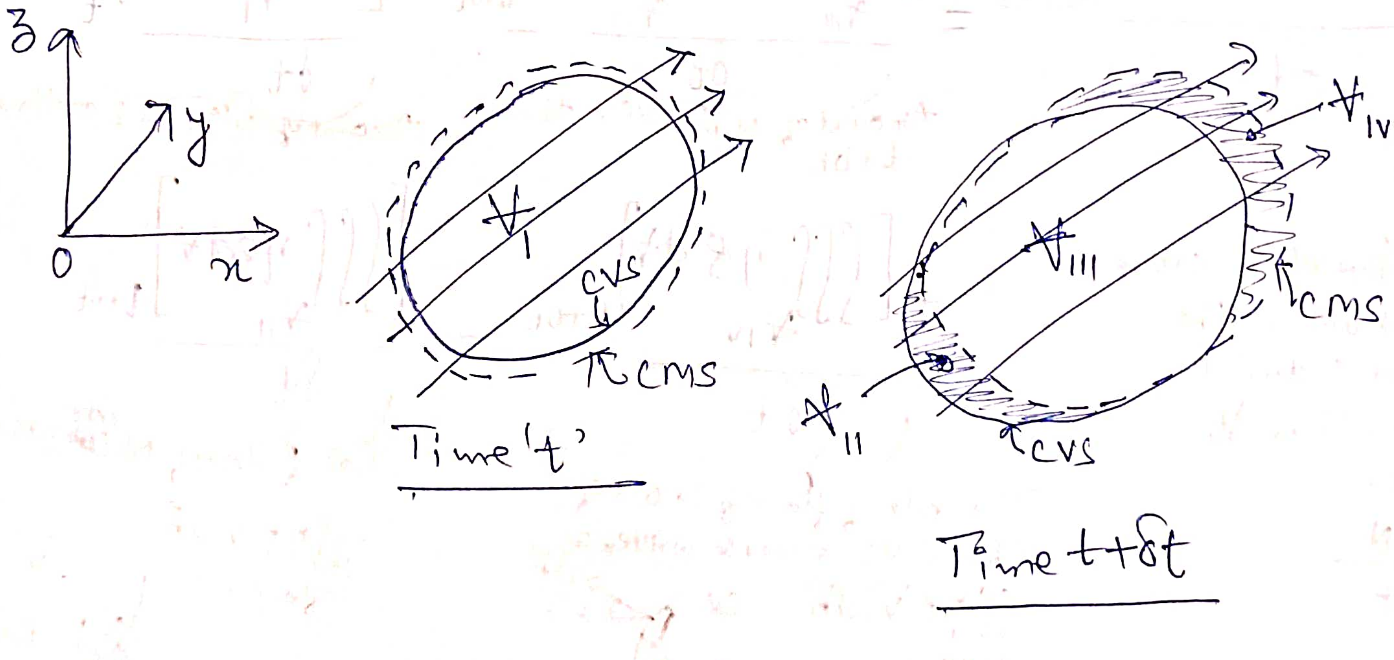
RTT (relates the control volume concept with that of control mass system)

Let N be the total amount of some property (mass, momentum, energy) within the control mass system at time t, and η be the amount of this property per unit mass throughout the fluid.



The RTT which is stated as "The time rate of increase of property N within a CMS is equal to the time rate of increase of property N within the CVS plus the net rate of efflux of the property N across the control surface.

$$\therefore \left(N_{t+\delta t} - N_t\right)_{CMS} = \underbrace{\left[\iiint_{V_{III}} \eta\rho\, dV + \iiint_{V_{IV}} \eta\rho\, dV\right]_{t+\delta t} - \left[\iiint_{V_{I}} \eta\rho\, dV\right]_t}$$

$\Downarrow$ add & subtract this term on RHS

$$+\left[\iiint_{V_{II}} \eta\rho\, dV\right]_{t+\delta t} - \left[\iiint_{V_{II}} \eta\rho\, dV\right]_{t+\delta t}$$

Dividing Overall eqn by δt

$$\frac{\left(N_{t+\delta t} - N_t\right)_{CMS}}{\delta t} = \underbrace{\frac{\left[\iiint_{V_{III}} \eta\rho\, dV + \iiint_{V_{II}} \eta\rho\, dV\right]_{t+\delta t}}{\delta t}}_{\text{Amount of N in CVS at time } t+\delta t} - \underbrace{\frac{\left[\iiint_{V_I} \eta\rho\, dV\right]_t}{\delta t}}_{\text{Amount of N in CVS at time } t}$$

$$+\underbrace{\frac{\left[\iiint_{V_{IV}} \eta\rho\, dV\right]_{t+\delta t}}{\delta t}}_{\text{Time rate of flow of N out of the CVS}} - \underbrace{\frac{\left[\iiint_{V_{II}} \eta\rho\, dV\right]_{t+\delta t}}{\delta t}}_{\text{Rate of flow of N into the CV}}$$

avg time rate of increase in N within the CMS during the time δt

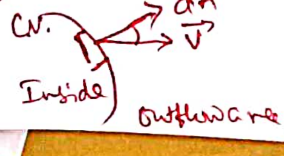
CV

$$\frac{dN}{dt}$$

Time rate of flow of N out of the CVS & may be written as

$\iint \eta\rho\, \vec{V}\cdot d\vec{A}$

outflow area



Rate of flow of N into the CV

$-\iint \eta\rho\, \vec{V}\cdot d\vec{A}$

inflow area

Sign Convention: The sign of $d\vec{A}$ is (+)ve if its direction outward normal

$$\therefore \boxed{\left(\frac{dN}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \eta \rho \, d\forall + \iint_{CS} \eta \rho \, \vec{V}_r \cdot d\vec{A}}$$

where V_r is ~~relative~~ fluid velocity relative to C.V. $\vec{V}_r = \vec{V} - \vec{V}_c$

$\vec{V}$ and $\vec{V}_c$ are velocity of fluid and C.V., as observed in a frame of reference (x,y,z).

Case(i): Continuity equation or Conservation of mass equation

Let $N = m$ (mass)

$$\eta = \frac{m}{m} = 1.$$

from RTT,

$$\left(\frac{dm}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} 1 \cdot \rho \, d\forall + \iint_{CS} 1 \, \rho \, \vec{V}_r \cdot d\vec{A}$$

$$= \frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \iint_{CS} \rho \, \vec{V}_r \cdot d\vec{A}$$

as per Gauss divergance theorem

$$\iint_{CS} \rho \vec{V}_r \cdot d\vec{A} = \iiint_{CV} \nabla \cdot \rho \vec{V}_r \, d\forall$$

$$\Rightarrow \left(\frac{dm}{dt}\right)_{CMS} = \frac{\partial}{\partial t}\iiint_{CV} \rho \, d\forall + \iiint_{CV} \nabla \cdot \rho \vec{V}_r \, d\forall$$

$$= \iiint_{CV} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V}_r\right) d\forall = 0$$ as per Continuity equ^n

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}_r) = 0}$$

for incompressible flow, ρ = Constant

$$\nabla \cdot (\rho \vec{V}_r) = 0$$

$$\Rightarrow \boxed{\nabla \cdot \vec{V}_r = 0}$$