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Functional Analysis.

Metric space :- A function from $X \cdot X$ to $\mathbb{R}^+ \cup \{0\}$ on any non-empty set X is called metric space if the following conditions are satisfied :-

(i). $d(x,y) \geq 0$

(ii). $d(x,y) = 0$ iff $x=y$

(iii). $d(x,y) = d(y,x)$

(iv). $d(x,y) \leq d(x,z) + d(z,y)$.

$$\boxed{d : \underbrace{X \times X}_{\text{Non-Empty set}} \longrightarrow \mathbb{R}^+ \cup \{0\}}$$

Non-Empty set

collection of well-defined object.

There will be no ambiguity or confusion by adding or removing any object in the collection.

EXAMPLES OF METRIC SPACE :-

usual metric on $\mathbb{R}$; $d(x,y) = |x-y|$

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Proof: $d(x,y) \geq 0$ is satisfied as
$|x-y| \geq 0$.

$d(x,y) = 0$ is satisfied as
$d(x,y) = |x-y| = 0$.
$\Rightarrow x = y$
$d(x,y) \Rightarrow |x-y|$
$= 0$.

$d(x,y) = d(y,x)$ is satisfied as
$|x-y| = |y-x|$.

$d(x,y) = |x-y|$
$= |x-y+z-z|$
$= |(x-z)+(z-y)|$.

$\int^{\circ}$: $|a+b| \leq |a|+|b|$.

$= |x-z| + |z-y|$

$d(x,y) = d(x,z) + d(z,y)$, satisfied.

$X = \mathbb{R}^2$.

$d(x,y) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.

Here, $x = (x_1, y_1)$
$y = (x_2, y_2)$.

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$(\mathbb{R}^2, d)$ & $(\mathbb{R}^2, d_1)$

- A metric space $\mathbb{R}^2$, called Euclidean plane is obtained if we take the set of ordered pairs of real numbers written as $x = \xi_1, \xi_2$
  $y = \eta_1, \eta_2$, etc.

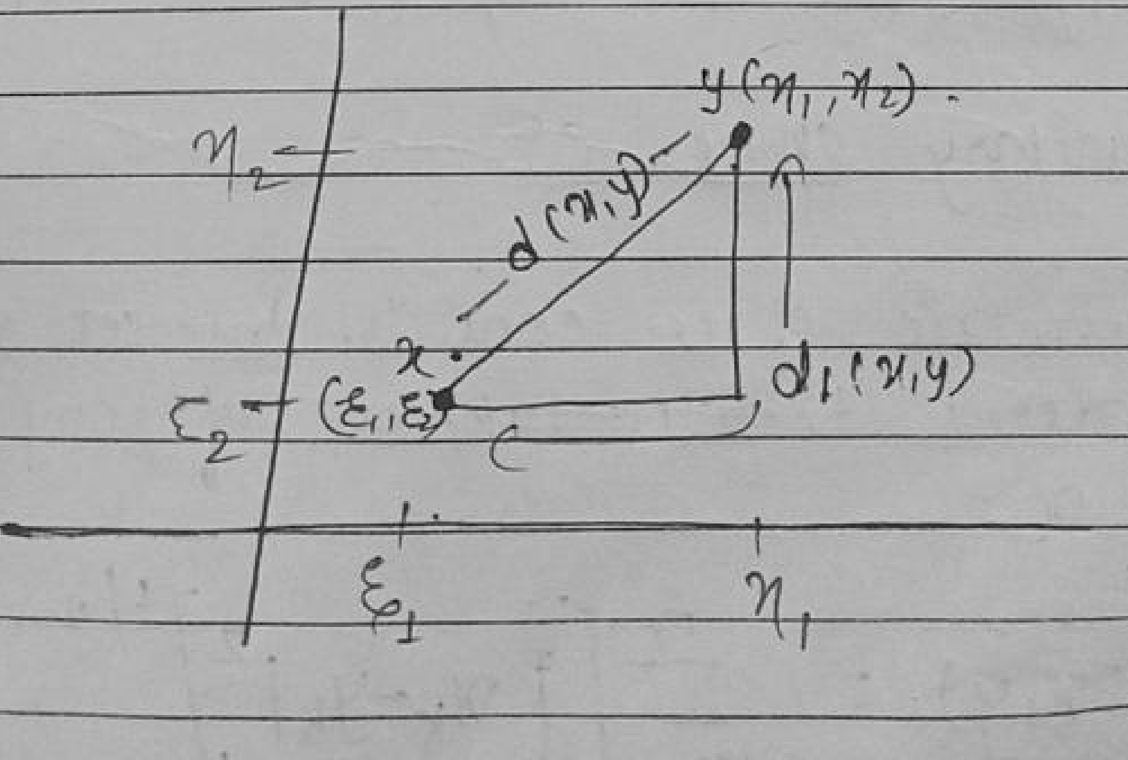
And an Euclidean metric define by

$$d(x,y) = \sqrt{(\xi_1-\eta_1)^2 + (\xi_2-\eta_2)^2}.$$

- Another metric is based, on obtaining, if we choose the same set $\mathbb{R}^2$ but another metric d_1, which is defined as

$$d_1(x,y) = |\xi_1-\eta_1| + |\xi_2-\eta_2|$$

d_1 is some times called the taxicab Metric.



Euclidean Space $\mathbb{R}^3$ —

A metric space consists the set of ordered triplets of real numbers

$x = (\xi_1, \xi_2, \xi_3)$.

$y = (\eta_1, \eta_2, \eta_3)$ etc. and the Euclidean Metric define by.

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2}$$

Euclidean space $\mathbb{R}^n$ —

A metric space consists the set of ordered n-tupples of real numbers.

$x = (\xi_1, \xi_2, ---, \xi_n)$.

$y = (\eta_1, \eta_2, ---, \eta_n)$. etc. and the Euclidean Metric define by -

$$d(x,y) = \sqrt{(\xi_1-\eta_1)^2 + (\xi_2-\eta_2)^2 + ---- + (\xi_n-\eta_n)^2}$$

Unitary space $\mathbb{C}^n$ —

A metric space consists the set of ordered ~~trip~~ n-tupples of complex no.

$x =$

$$d(x,y) = \sum_{k=1}^{n} \left[|x_k - y_k|^2 \right]^{1/2}.$$

Sequence Space l^{∞}

A set X taken as the set of all bounded sequence of complex no. then is every element of X is a complex sequence such that $\forall$ $x = (\xi_j)$ $\forall$ $j = 1, 2, \ldots$

$x \in l^{\infty}$ where, $x = (\xi_1, \xi_2, \ldots)$.

we have $|\xi_j| \leq C_x$

where, $C_x \rightarrow$ Real no. which may depend upon x but does not depend on j.

we now choose a metric define by

$$\rightarrow d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|$$

Function Space $C[a,b]$.

If we take X as the set of all real valued functions $(x, y, \ldots)$ which are functions of an independent real variable d and define as well as continuous on a given closed interval $[a, b]$.

$$x : [a,b] \longrightarrow \mathbb{R}$$

$$x(t) \in \mathbb{R}, \quad t \in [a,b].$$

continuous on $[a, b]$

Choosing the metric defined by

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

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Discrete Metric Space —

If we take any set S and, we define a metric (X, d) and $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

then this space (X, d) is called a discrete metric space.

It rarely occurs in applications. However, we shall use it in examples for illustrating certain concepts.

Ex: →
$d(x,y) = (x-y)^2$
→ $d(x,y) = \sqrt{|x-y|}$] $X = \mathbb{R}$
→ $d(x,y) = \int_a^b |x(t) - y(t)|\, dt$
{ $X = C[a,b]$ }

Sequence space s

A set of all sequences either bounded or unbounded of complex or real no. and Metric (d) defined by — Frechet Metric.

Imp

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

where, $x = \xi_j$

$y = \eta_j$

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Note that a metric in the previous example (l^∞, d).

$$\longrightarrow \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| = d(x,y).$$

would not be suitable in the present case because metric would not be remain finite.

Proof of:- $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$.

$\rightarrow$ $d(x,y) \geq 0$.

$\Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \geq 0$.

$\rightarrow$ $d(x,y) = 0$ iff $x = y$

$\Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \xi_j|}{1+|\xi_j - \eta_j|} = 0$.

$\rightarrow$ $d(x,y) = d(y,x)$.

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\eta_j - \xi_j|}{1+|\eta_j - \xi_j|}.$$

$\rightarrow$ for satisfying (iv) condition. we use auxiliary function define on $\mathbb{R}$ by $f(t) = \frac{1}{1+t}$.

$$f'(t) = \frac{1}{(1+t)^2} \geq 0.$$

Hence, f is monotonically increasing

consequently, $|a+b| \leq |a|+|b|$.

$$f[|a+b|] \leq f(|a|+|b|)$$

$$\frac{1}{1+|a+b|} \leq \frac{1}{1+|a|+|b|}$$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}$$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|}$$

$$\left\{ \because 1+|a|+|b| > 1+|a| \right.$$

$$\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Now we put in this inequality

$$a = \xi_i - C_i.$$

$$b = C_j - \eta_j.$$

$$\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \frac{|\xi_i - C_i|}{1+|\xi_i - C_i|} + \frac{|C_j - \eta_j|}{1+|C_j - \eta_j|}$$

Now, $x = (\xi_i)$:

$y = \eta_i$.

$z = C_{ij}$.

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we multiply both sides $\frac{1}{2^q}$ &

take summation from 1 to ∞.

$$\sum_{q=1}^{\infty} \frac{1}{2^q} \frac{|x-y|}{1+|x-y|} \leq \sum_{q=1}^{\infty} \frac{1}{2^q} \left[\frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \right]$$

$$\boxed{d(x,y) \leq d(x,z) + d(z,y)}$$

~~Example~~: space of Bounded function. B(A)

By definition,

The set $X = B(A)$ with the metric (d) defined by $d(x,y) = \sup_{t \in A} |x(t) - y(t)|$

$x(t) - y(t)$ forms a metric space where $x(t)$ & $y(t) \in B(A)$ & are functions defined & bounded. On a given set (A)

Sequence space l^p, Hilbert sequence space l^2, Holder & Minkowski inequalities for sum.

Let $p \geq 1$ be a fixed real no.
by definition in each element in l^p in the sequence. $x = (\epsilon_j)$ of no.
s.t.

$$|\epsilon_1|^p + |\epsilon_2|^p + \cdots \text{ : converges.}$$

$$\boxed{\sum_{j=1}^{\infty} |\epsilon_j|^p < \infty}$$

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Now we define a metric by
$$d(x,y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p\right)^{1/p} \quad \text{—— } (*)$$

where, $x = \xi_i \in l^p$ & $y = \eta_i \in l^p$.

Now, we prove that l^p is a metric space

Clearly, the given metric $(*)$ satisfies the I^{st} 3 axioms ~~of metric~~ provided that the series on R.H.S converges.
To prove that it does converge and the IV axiom will be satisfied.
proceeding step-wise we shall give (a) $\rightarrow$ an auxiliary inequality
(b) $\rightarrow$ the Holder inequality from (a)
(c) $\rightarrow$ the Minkowski " " from (b).
(d) $\rightarrow$ the triangular " " from (c)

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Proof:-

(a). Let $p > 1$ & define q by $\frac{1}{p} + \frac{1}{q} = 1$ —— (i)

p & q are then called conjugate exponents.
Now,

$$\frac{p+q}{pq} = 1.$$

$$p+q = pq.$$

$$(p-1)(q-1) = 1$$

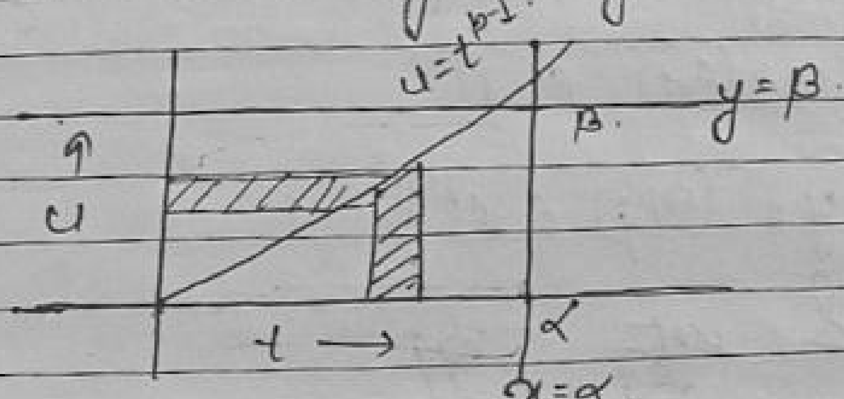
Hence, $\frac{1}{p-1} = q-1.$

So that,

$$u = t^{p-1}$$

$$t = u^{q-1} \quad \because \; u^{\frac{1}{p-1}} = t$$

Let α & β be positive numbers. Since $\alpha \cdot \beta$ is the area of the rectangle we thus obtain by integration by inequality



$$\alpha\beta \le \int_0^{\alpha} t^{p-1}\,dt + \int_0^{\beta} u^{q-1}\,du$$

$$= \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

). Let $(\bar{\xi}_j)$ & $(\bar{\eta}_j)$ be such that —

$$\sum |\bar{\xi}_j|^p = 1, \quad \sum |\bar{\eta}_j|^q = 1 \longrightarrow (7)$$

Setting $\alpha = |\bar{\xi}_j|$ & $\beta = |\bar{\eta}_j|$

in previous inequality, we get

$$|\bar{\xi}_j \bar{\eta}_j| \le \frac{1}{p} |\bar{\xi}_j|^k + \frac{1}{q} |\bar{\eta}_j|^q$$

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By summation we sum over i, we have

$$\sum |\bar{\xi}_i \bar{\eta}_i| \leq \frac{1}{p} + \frac{1}{q} = 1 \quad \text{(8)}$$

Now, we take any non-zero sequence

$x = (\xi_i) \in l^p$

$y = (\eta_i) \in l^p$

& set:

$$\left.\begin{aligned} \bar{\xi}_i &= \frac{\xi_i}{\left[\sum |\xi_k|^p\right]^{1/p}} \\ \bar{\eta}_i &= \frac{\eta_i}{\left[\sum |\eta_m|^q\right]^{1/q}} \end{aligned}\right\} \text{(9)}$$

Then it is obvious that (7) is satisfied by eq (9).

$$\bar{\xi}_i = \sum |\bar{\xi}_i|^p = \cancel{\frac{\sum |\xi_i|^p}{\left[\sum |\xi_k|^p\right]^{1/p}}} \left| \frac{\xi_i}{\left(\sum |\xi_k|^p\right)^{1/p}} \right|$$

$$= \frac{\sum_{i=1}^{\infty} \xi_i^p}{\sum_{k=1}^{\infty} |\xi_k|^p}$$

$$= 1.$$

HOLDER INEQUALITY