$S_n \rightarrow$ set of first integers less than n

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Set :- Collection of well defined objects
(There will be no ambiguity or confusion while adding or removing objects in the collection).

Ex:- Collection of all batsmen whose avg. is below 50.
collection of real no. | complex no. | Q / 8 |
Transcedental no. | algebraic | conti. fns | binomials |
Integers | natural no.

Algebraic no :- that no. which are root of any polynomial with rational coeffo. otherwise that no. is called transcedental no.

$\alpha \in \mathbb{C}$
$P_n(\alpha) = 0$
$a_i's \in Q$

Power set :- set of all subsets of any set is called power set of ~~all~~ the sets.

Cardinality :- no. of elements in a set

Countable set :-

Finite set :- $S_n = \{1, 2, \dots n\}$ $\exists f$ bijective
$f: A \rightarrow S_n$ $S_n \rightarrow A$ for $n \in \mathbb{N}$

Similar set :- Two sets A & B are said to be similar if $\exists$ a 1-1 correspondance b/w these 2 sets.
Sets A & B are said to have same cardinality or same no. of elements or to be equipotent $A \approx B$ if there is a fn $f: A \rightarrow B$ which is bijective that is both 1-1 & onto.

Ex:- let A & B be sets with exactly three elements
$A = \{2, 3, 5\}$

$B = \{$ Marc, Eric, Peterson $\}$
then we can find 1-1 correspondance b/w A & B we can label the element A as the first element, second element & third element And label B similarly then the rule which pairs the first element A & B, pairs 2nd element of A & B & pairs 3rd element of A & B i.e the fn $f: A \to B$

$f(2) =$ Marc
$f(3) =$ Eric
$f(5) =$ Peterson

Hence A & B are equipotent

Ex:- Let $I = [0,1]$ & S = any other closed interval say, $[a,b]$ where $a < b$

$f(x) = (b-a)x + a$
$f(x_1) = (b-a)x_1 + a$ $\quad x_1, x_2 \in I$ s.t $f(x_1) = f(x_2)$
$f(x_2) = (b-a)x_2 + a$
$(b-a)x_1 + a = (b-a)x_2 + a$
$x_1 = x_2$
$\therefore$ 1-1

To prove onto
$f(x) = y \quad , y \in S$
$(b-a)x + a = y$
$y - a = (b-a)x$
$$x = \frac{y-a}{b-a}$$

$\therefore$ sets are equipotent
Any two closed interval have the same cardinality

Q2 $\quad I = (0,1)$

Ex:- E ⊂ P

E = {2, 4, 6 ... } Pupn

P ≃ E → denumerable though infinite

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S = any other closed interval say [a, b] where $a < b$

19-7-23 Ex:- $\pi, \pi^e, e^\pi, \sqrt{2}$ are transcedental no.

* All irr. are not transcedental

Ex :- $\sqrt{2}$ is " "

Def:- A set S is infinite if it has the same cardinality has a proper subset of itself o/w S is finite.

always any 2 closed interval $[0,1] \cong [a,b]$

$[1/2, 1/4]$ cardinality same

Ex:- N, Z, Q, R

Ex:- Consider any 2 set A & B. Any 2 set let

$A' = A \times \{1\}$

$B' = B \times \{2\}$

then A' is equipotent to A $(A' \cong A)$

then B' ≅ " B

$f: A \to A'$, $f(a) = (a, 1)$

$f: B \to B'$, $f(b) = (b, 2)$

To prove:- A' & A are equipotent be define a fn $f: A \to A'$ s.t $f(a) = (a, 1)$ here we can easily see that this fn f is one-one & onto (bijective) Hence set A & A' are equipotent (By def.) similarly for B & B'

Denumerable set & countable set / countably infinite :- A set D is said to be denumerable or countably infinite if D has the same cardinality as set of N no.

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Countable set:-

A set is ***countable*** if it is ***finite*** or ***denumerable*** (enumerable)
& a set is non denumerable if it's not countable.

Ex:- Any infinite seq $a_1, a_2, \dots$ of distinct element is countably infinite.

$f : N \to A$

$A = \{a_1, a_2, \dots a_n\}$

$f(n) = a_n$

$f : N \to A$ is bijective (in exam prove)

Cardinality of N & A are eq.

$\therefore$ Denumerable

Ex:- $\{(1,1), (4,8), (9,27), \dots (n^2, n^3) \dots\}$, where $(1,1)$, $(4,8)$, $(9,27)$, …, (n^2, n^3) are marked 1, 2, 3, …, n

distinct elements

denumerable

Ex:- Consider the product $N \times N$. The set $N \times N$ can be written as an infinite seq. as follows $\{(1,1), (2,1), (1,2), (1,3), (2,2), (3,1), \dots\}$

This seq. can be determined as follows

$(1,1)$, $(1,2)$, $(1,3)$ ...
$(2,1)$, $(2,2)$, $(2,3)$ ...
$(3,1)$, $(3,2)$, $(3,3)$ ...

$f(1) = (1,1)$
$f(2) = (2,1)$
$f(3) = (1,2)$
$f(4) = (1,3)$

Set
- countable
  - finite
  - denumerable | countably infinite
- non-denumerable | un-countable

$\mathbb{N}\times\mathbb{N}\times\mathbb{N}$

$\mathbb{N}\times\mathbb{N}\approx\mathbb{N}$

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Ex: Q) Prove that the set $A = \mathbb{N}\cup\{0\}$ & set $A\times A$ has same cardinality as set of $\mathbb{N}$ no.

Sol^n:- ~~|A|~~

let $f: \mathbb{N}\to A$

$f(n) = n-1$

$n_1, n_2 \in \mathbb{N}$

$n_1 - 1 = n_2 - 1$

$n_1 = n_2$

$\therefore$ 1-1

$f(n) = y$ $\quad n\in\mathbb{N}$

$y = n-1$ $\quad y\in A$

$n = y+1$.

$\therefore$ onto

$f: N\to A$ is bijective

cardinality eq

$\therefore$ equivalent

To prove:- $|\mathbb{N}|$ & $|A\times A|$ are same

let $f: N\to A\times A$

$f(n) = \cancel{2r}\ 2^r(2s+1) = (r,s)$

$1\to(0,0)$

$2\to(1,0)$

$3\to(0,1)$

$\vdots \quad \cdots \quad \vdots$

$f: N\to A\times A$ is bijective

cardinality same

$f(n) = (r,s)$ s.t $n = 2^r(2s+1)$

1) Theorem:- Every infinite set contains a subset which is denumerable. (Den/Non Den ?)

2) Theorem:- A subset of Denumerable set is finite or denumerable.

Corollary:- A subset of a countable set is countable.

3) Theorem:- Let $A_1, A_2, A_3, \ldots$ be a seq. of pairwise disjoint denumerable sets then

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the $\cup A_i$, $i \in \mathbb{N}$ is denumerable.

Corollary :- A countable union of countable set is countable. Ex!- $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q}$

Pf: 1) Let $f : \underset{\text{family of subsets}}{P(A)} \longrightarrow A$ be a choice function
consider the following seq.

$a_1 = f(A)$ → except $A \longrightarrow$ infinite set
$a_2 = f(A \setminus \{a_1\})$ $D \longrightarrow$ denumerable set
$a_3 = f(A \setminus \{a_1, a_2\})$
— — — — — — —
$a_n = f(A \setminus \{a_1, a_2, \ldots, a_n\})$

Since A is infinite, from set $A \setminus \{a_1, a_2, \ldots, a_{n-1}$ is non empty for every natural no. $\mathbb{N}$
Furthermore since f is a choice fn
$a_n \neq a_i$ for $i < n$
Thus the a_n are distinct & therefore ~~$D = A$~~.
$D = \{a_1, a_2, \ldots\}$ is denumerable subset of A
~~Essentio~~ Sincerely choice fn f chooses an element $a_1 \in A$ then chooses an element a_2 from the elements which remain in A & so on. Since A is infinite the set of element which remain in A is non empty.

Pf 2:- consider any denumerable set say
$A = \{a_1, a_2, \ldots\}$ & ——— (i)
let B be a subset of A. If $B = \phi$ then it is obviously finite.
Suppose $B \neq \phi$

let b_1 be the first element in seq. in (ii) s.t $b_1 \in B$. Let b_2 be the first element which follows b_1 in the seq. in (ii) s.t $b_2 \in B$ & so on. Then $B = \{b_1, b_2, \ldots\}$ if the seq. $b_1, b_2 \ldots b_n$ ends. Hence B is finite otherwise B is denumerable.

Pf 3) Let $A_1, A_2, \ldots$ be a seq. of pairwise disjoint denumerable set then we have to prove $S = \cup A_i, i \in \mathbb{N}$ is denumerable.

Suppose $A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$

define $D_n = \{a_{ij} : i+j = n, n > 1\}$

For ex., $D_2 = \{a_{11}\}$, $D_3 = \{a_{12}, a_{21}\}$.

$D_4 = \{a_{13}, a_{31}, a_{22}\}$

note that each D_n is finite infact D_n has $n-1$ units. Hence by following article $T = \cup(D_j : j > 1)$ is countable on the other hand the union of finite D's is the same as the union of A's i.e $T = S$

Thus S is countable

(A) Article:- let $C = \{S_i : i \in \mathbb{N}\}$ be a countable collection of finite set & let $C^* = \cup_i S_i$

If C^* is empty then C is countable.

Suppose ~~C do~~ $C^* \neq \phi$ can define

$A_1 = S_1$

$A_2 = S_2 \setminus S_1$

$A_3 = S_3 \setminus S_2$ & so on

then the set A_i's are finite & pairwise disjoint say $A_1 = \{a_{11}, a_{12}, \dots, a_{1n_1}\}$
$A_2 = \{a_{21}, a_{22}, \dots, a_{2n_2}\}$

then union $B = \bigcup_i A_i$ can be written as a seq. of as follows $B = \{a_{11}, a_{12}, \dots, a_{1m}$
$a_{21}, a_{2} \dots a_{2m}$

i.e first we write down elements of $A_1, A_2 \dots$

we have to prove that set B is countable so we define a fn $f : B \to \mathbb{N}$ as follows $f(a_{ij}) = n_1 + n_2 + \dots + n_{i-1} + j$

clearly f is bijective.

Hence B is countable.

However B is also the union of sets in C i.e $B = C^*$

$\therefore$ C^* is countable

Cicle → Statement :- A countable union of finite sets is countable.

Imp
-7-23 Theorem :- The set of all Rational no. is denumerable / countably infinite.

Pf :- $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$

$\mathbb{Q}^+ \Rightarrow$ set of all +ve rational no.

$\mathbb{Q}^- \to$ " " " -ve " " respectively

Now let a $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ s.t

$f(p/q) = (p, q)$ where $\frac{p}{q}$ is any

$(p,q)=1$

element of $\mathbb{Q}^+$ expressed as the ratio of two relatively prime +ve integers
gcd = 1

$\frac{1}{q} < 1 \quad q \in N \quad f(\frac{p}{q}) = (-p, q)$

$f : Q^- \to N \times N$

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then f is bijective map hence Q^+ has same cardinality.

one to one map

f so Q^- has same cardinality as a subset of $N \times N$ since $N \times N$ is denumerable so by theorem 2 subset of " " either finite or denumerable since Q^+ is infinite set so it's equipotent subset $N \times N$ must be denumerable.

Similarly ~~being~~ we'll prove Q^-.

Since countable union of denumerable set is denumerable.

hence the set Q is denumerable.

Note

\# i) If a fn from $A \to B$ ($f : A \to B$) is 1-1 then the cardinality of A must be less than or eq. to cardinality of B & $\exists$ a subset in B which has the same cardinality as A

ii) If a $f : A \to B$ is onto then cardinality of B must be ~~or~~ less than or eq. to " " A & $\exists$ a subset in A which as same cardinality as B.

→ Not every infinite set is countable.

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Theorem:- Prove that unit interval $[0,1]$ is non-denumerable or uncountable.

Def → A set A is said to have the <u>power of continuum</u> if A has the same cardinality as the unit interval $[0,1]$.
Ex. $[0,b)$, $[a,b]$, (a,b) also have POC

Pf:- Assume $[0,1]$ to be denumerable then
$[0,1] = I = \{x_1, x_2, x_3, \dots\}$
Now each element in I can be written in the form of an infinite decimal as follows:-

$x_1 = 0.a_{11}a_{12}a_{13}\dots a_{1n}\dots$

$x_2 = 0.a_{21}a_{22}a_{23}\dots a_{2n}\dots$

– – – – – –

– – – – – – –

$x_n = 0.a_{n1}a_{n2}\dots a_{nn}\dots$

where $a_{ij} \in \{0,1,2,\dots 9\}$ & where each decimal contains infinite no. of non zero elements.
Then we write ~~one~~ 1 as $0.999\dots$ & for those no. which can be written in the form of a decimal in two ways for ex:-

$1/2 = 0.50000\dots$

$= 0.4999\dots$ (infinite no. of non zero elements)

(in one of them there is an infinite no. of lines & in the other all es finite set of digits are zero)

we write the infinite decimal in which an infinite no. of lines appear.

(Now construct real no $y = 0.b_1b_2\dots b_n\dots$

NO

which will belong to I in the following way.
choose b_1 so that $b_1 \neq a_{11}$ & $b_1 \neq 0$.
choose b_2 " " $b_2 \neq a_{22}$ & $b_2 \neq 0$
& so on . . . . .

Note $y \neq x_1$ since $b_1 \neq a_{11}$.
similarly $y \neq x_2$ " $b_2 \neq a_{12}$
& so on . . .
i.e $y \neq x_n \quad \forall \; n \in \mathbb{N}$
Thus $y \neq I$ which contradicts the fact that $y \subset I$
Hence I is non denumerable.

Note:- Prove that the set of R no. also has the power of continuum

Pf:- consider $f: R \to D$ where D is $(-1,1)$

$(-1,1) \approx [0,1]$

$$f(x) = \frac{x}{1+|x|}$$

By horizontal line test
f(x) is bijective

$R \approx (-1,1)$
↓
non denumerable

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Prove that the closed interval $[0,1] \approx (0,1)$ are equipotent.

ii) $[0,1] \approx [0,1)$ are equipotent

iii) $[0,1] \approx (0,1]$ " " .

i) consider the $f: [0,1] \to (0,1)$ s.t

$$f(x) = \begin{cases} 1/2 & \text{if } x=0 \\ 1/n+2 & \text{if } x = 1/n \;, n \in N \\ x & \text{if } x \neq 0, 1/n \end{cases}$$

Hence f is bijective

$\Rightarrow [0,1] \approx (0,1)$

ii) Consider the $f: [0,1] \approx [0,1)$ s.t

$$f(x) = \begin{cases} 1/n+1 \;, & \text{if } x = 1/n \;, n \in N \\ x \;, & \text{if } x \neq 1/n \;, n \in N \end{cases}$$

Hence f is bijective

$\Rightarrow [0,1] \approx [0,1)$

iii) Consider the $f: [0,1] \approx (0,1]$ s.t

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n+1 & \text{if } x = 1/n \\ x & \text{if } x \neq 0 \end{cases}$$

or

$f(x) = 1 - x$

Hence f is bijective

$[0,1] \approx (0,1]$

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Prove:- If A, B, C are any set then i) A is equipotent to A
ii) If $A \approx B$ then $B \approx A$
iii) If $A \approx B$ & $B \approx C$
then $A \approx C$

Pf:- i) To proof:- $A \approx A$
~~let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$~~
~~let $B = \{b_1, b_2, b_3, \ldots, b_n\}$~~
~~let $C = \{c_1, c_2, c_3, \ldots, c_n\}$~~
~~To show A is bijective~~ $|A| = |A|$
Consider the $f: A \rightarrow A$ s.t $f(x) = x$
f is bijective
$\Rightarrow A \approx A$

ii) Given $A \approx B$
To show $B \approx A$
Pf:- $\exists$ ~~consider~~ the $f: A \rightarrow B$ ~~s.t~~ which is bijective.
$\therefore f^{-1}: B \rightarrow A$
f^{-1} is bijective
$\Rightarrow B \approx A$

iii) Given $A \approx B$, $B \approx C$
To show $A \approx C$
If $A \approx B$, $\exists$ a $f_1: A \rightarrow B$ which is bijective
If $B \approx C$, $\exists$ a $f_2: B \rightarrow C$ which is bijective
By composition of function
$f_2 \circ f_1 : A \longrightarrow C$
Hence $f_2 \circ f_1$ is bijective
cardinality will be same
$\therefore$ Both are equipotent i.e $A \approx C$

★★ 1. If f is one-one only & function g is onto only then composition of mapping gof is Bijective

Cardinal Numbers :- Frequently we want to know the size of a given set without necessarily comparing two with another set.

For finite set there is no difficulty but on the other hand for infinite sets it is not sufficient to just say that the set has infinitely many elements since not all infinite sets are equipotent. To solve this problem we introduce concept of cardinal numbers.

Each set A assign a symbol in such a way that 2 sets A & B are assign the same symbol iff they are equipotent. This symbol is called the cardinality or cardinal no. of A & denoted by $|A|$ or $n(A)$ or $card(A)$

Q. Prove that the following intervals have power of continuum.

$[a,b]$
(a,b)
$[a,b)$
$(a,b]$

cardinality = c

If we define $f(x) = a + (b-a)x$
is b/w each pair of sets i) $[0,1]$ & $[a,b]$

A is infinite ⇒ any bijective mapping from $A \to N$, $N \to A$

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ii) $(0,1)$ & (a,b)
iii) $[0,1)$ & $[a,b)$
iv) $(0,1]$ & $(a,b]$

26-7-23 Finite cardinal Number:- ~~The~~ Two obvious ~~signal~~ symbol are used for the cardinal numbers of finite sets. i.e 0 is assigned to empty set and n is assigned to the set, $A = \{1, 2, \ldots n\}$, $|A| = n$

$A = \phi$, $|A| = 0$
$A = \{\phi\}$, $|A| = 1$
$A = \{\phi, \{\phi\}\}$, $|A| = 2$
$A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$ $|A| = 3$

Although the $\mathbb{N}$ no. & the cardinal no. n are technically different things. There is no conflict using same ~~signal~~ symbol in these two rules. The ~~fn~~ cardinal no. of finite sets are called finite cardinal no.

$|A| = n$ $|A| = 3$ iff $A \approx \{1,2,3\}$
iff $A \approx S_n = \{1, 2, \ldots n\}$ $|A| = 2$ iff $A \approx \{1,2\}$

Transfinite cardinal no.:- cardinal no. of infinite sets are called transfinite cardinal no. The cardinal no. of infinite set $\mathbb{N}$ is $\aleph_0$. (aleph n...) This notation is introduced by Cantor & the symbol $\aleph_0$ is the first letter Aleph of ~~de~~ Hebrew alphabet. Thus $|A| = \aleph_0$ iff A is equipotent to $\mathbb{N}$ $(A \simeq \mathbb{N})$, In particular $|\mathbb{Z}| = \aleph_0$, $|\mathbb{Q}| = \aleph_0$ (Because are countable)

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The cardinal no. of unit interval $I = [0,1]$ is denoted by C (continum) & it is called power of continum. Thus $|A| = C$ iff $A \cong [0,1]$. In particular we have $|R| = C$ & cardinal no. of any interval is C. Ex: $|(2,3)|$, $|[-1,5]| = C \cong [0,1]$

→ onto (opposite)

Ordering of cardinal No:- Let A & B be sets. We say that $|A| \leq |B|$ if A has same cardinality has a subset of B equivalently if $\exists$ one-one (injective) function $\boxed{f: A \to B}$

Ex:- Let n be a finite cardinal then $n < \aleph_0$ since any " set is equipotent to a subset $\mathbb{N}$. Thus we may write $0 < 1 < 2 < 3 \ldots < n < \ldots < \aleph_0$

$f: A \to B$
(subset) $B \subseteq N$
(injective)

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Ex:- Consider set $\mathbb{N}$ of natural no. & unit interval $I = [0,1]$ then the $f: \mathbb{N} \to [0,1]$ s.t

$$f(n) = \frac{1}{n}$$

If we take any n_1 & $n_2 \in \mathbb{N}$

$$f(n_1) = f(n_2)$$

$$\frac{1}{n_1} = \frac{1}{n_2}$$

$$n_2 = n_1$$

$\Rightarrow f$ is one-one

$\therefore \; |\mathbb{N}| \leq [0,1]$

$\mathbb{N} = \{1, 2, 3, \ldots\}$
$B = \{2, 4, 6, \ldots\}$
$B \subseteq \mathbb{N}$ $|B| = \aleph_0$, $|\mathbb{N}| = \aleph_0$

Since $\mathbb{N}$ is countable & $[0,1]$ is uncountable/non-denumerable. Hence $|\mathbb{N}| \neq [0,1]$
$\Rightarrow$ $|\mathbb{N}| < |[0,1]|$
$\Rightarrow$ $\aleph_0 < c$

Ex:- Let A be any infinite set then A contains a subset which is denumerable.
Accordingly for any infinite set A we always have $\aleph_0 \leq |A|$

Q?

Cantor's Theorem finite or infinite

Power set of A

For any set A we have $|A| < |P(A)|$ or for any cardinal no. α, we have $\alpha < 2^{\alpha}$

can be finite or transfinite

$|\mathbb{N}| = \aleph_0$
$P(\mathbb{N}) = 2^{\aleph_0}$
$\aleph_0 < 2^{\aleph_0} \leq c < 2^c$

Power set $= 2^n$

Pf:- The $g : A \to P(A)$ defined by $g(a) = \{a\}$ is a 1-1 mapping then
$|A| \leq |P(A)|$

If we now show that $|A| \neq |P(A)|$ then the theorem follows.
Suppose on contrary $|A| = |P(A)|$
$\exists\ f : A \to P(A)$
s.t f is bijective
Let $a \in A$ be called a "bad" element if a is not a member of set which is its image
i.e $a \notin f(a)$ then 'a' is called "bad" element
Now let B be the set of bad elements i.e
$B = \{x \in A : x \notin f(x)\}$

$b \in A$ $\quad$ $b \notin D$ (bad element)
$\{b\} \in P(A)$ $\quad$ $b \in D$ (good element)
$D \subset A$
$f(b) = D$

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Now $B \subset A$, hence $B \in P(A)$.
Since f is onto, hence $\exists$ a element $b \in A$ s.t.
$f(b) = B$
If $b \in B$ then by definition of B, $b \notin f(b) = B$
which is impossible likewise if $b \notin B$
then $b \in f(b) = B$ which is also impossible.
Thus original assumption that $|A| = |P(A)|$ has
led to a contradiction. Hence our assumption
is false. So the theorem is true.

a) Schroeder - Bernstein Theorem:-
If $|A| \leq |B|$ & $|B| \leq |A|$ then $|A| = |B|$

b) Theorem:- Let X, Y, X_1 be sets such that
$X \supseteq Y \supseteq X_1$ & $X \simeq X_1$ then $X \simeq Y$

Law of trichotomy:- For any two sets A & B,
exactly one of the following is true
$|A| < |B|$, $|A| = |B|$, $|A| > |B|$

Imp Continum hypothesis:- By cantor's theorem $\aleph_0 < 2^{\aleph_0}$
and we also know that by previous theorem
$\aleph_0 < C$. continum hypothesis tells us relationship
b/w $2^{\aleph_0}$ & C.

Pf ?? prove that there is no transfinite no. b/w $2^{\aleph_0}$ & C

Theorem Statement 1 $\longrightarrow$ $2^{\aleph_0} = C$
Statement 2 $\longrightarrow$ $\exists$ no cardinal no. ~~be the~~
β s.t. $\aleph_0 < \beta < C$

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domain small

Pf:- (a, b) Since $X \simeq X_1$ $\exists$ bijective map $f: X \to X_1$
for both Since $X \supseteq Y$ the restriction of f to Y,
which also denoted by f is also 1-1.

Subset can be 1-1 only

let $f(Y) = Y_1$ then Y & Y_1 are equipotent means $X \supseteq Y \supseteq X_1 \supseteq Y_1$ & $f: Y \longrightarrow Y_1$ is bijective but now $Y \supseteq X_1 \supseteq Y_1$ & $Y \simeq Y_1$.

For similar reasons X_1 & $f(X_1) = X_2$ are equipotent, $X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2$

$f: X_1 \longrightarrow X_2$ is bijective.

Accordingly $\exists$ equipotent sets $X, X_1, X_2, \ldots$ & equipotent sets $Y, Y_1, Y_2, \ldots$ s.t

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq X_3 \ldots$$

& $f: X_k \longrightarrow X_{k+1}$ is bijective

$f: Y_k \longrightarrow Y_{k+1}$ is also bijective

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2$

Then $X = (X \setminus Y) \cup (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup (Y_1 \setminus X_2) \ldots \cup B$

$Y = (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup (Y_1 \setminus X_2) \cup \ldots \cup B$

Furthermore $(X \setminus Y), (X_1 \setminus Y_1), (X_2 \setminus Y_2) \ldots (X_2 \setminus Y_2)$ are equipotent.

In fact $f: (X_k \setminus Y_k) \longrightarrow (X_{k+1} \setminus Y_{k+1})$ is bijective.

Consider the function $g: X \longrightarrow Y$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_k \setminus Y_k \text{ or } x \in X \setminus Y \\ x & \text{if } x \in Y_k \setminus X_k \text{ or } x \in B \end{cases}$$

then g is 1-1 & onto

$\therefore X \simeq Y$

Pf (c) Let R be the set of Real no. & $P(Q)$ be the power set of Rational no.

Furthermore let the function $f: \mathbb{R} \longrightarrow P(Q)$ be defined by $f(a) = \{x : x \in Q, x < a\}$

Now we shall show that f is 1-1

let $a, b \in R$, $a \neq b$ say $a < b$

$(-\infty, a) \cap \mathbb{Q}$

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By property of Real no. $\exists$ a rational no. s.t

$a < x < b$

then $x \in f(b)$ $\quad (-\infty, b) \cap \mathbb{Q}$

& $x \notin f(a)$ $\quad (-\infty, a) \cap \mathbb{Q}$

$(a \neq b)$

So $f(a) \neq f(b)$

$\Rightarrow$ f is 1-1

$\mathbb{Q} \cong \mathbb{N}$

Thus $|\mathbb{R}| \leq |P(\mathbb{Q})|$

$|\mathbb{Q}| = \aleph_0$

Since $|\mathbb{R}|$ $\quad c \leq 2^{\aleph_0}$

$|P(\mathbb{Q})| = 2^{\aleph_0}$

Now let $e(P)$ be $C(N)$ be the family of characteristic function

$$f_A : X \to \{0, 1\}$$

$$f_A(x) = \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases}$$

Now let N be the f

$$f : N \to \{0, 1\}$$

which is equivalent to $P(\mathbb{N})$

Now let $I = [0, 1]$ & let $f^* : C(\mathbb{N}) \to I$ be defined by $f^*(g) = 0 \cdot f(1) f(2) f(3) \ldots$ & infinite decimal consist of zeroes & ones.

Suppose $f, g^* \in C(\mathbb{N})$ & $f \neq g^*$ then decimal would be different so $f^*(f) \neq f^*(g)$.

$\Rightarrow$ f^* is one-one

$$|C(N)| \leq |[0, 1]|$$

$$2^{\aleph_0} \leq c$$

Now we have proven that $2^{\aleph_0} \leq c$ &

$c \leq 2^{\aleph_0}$

$$\Rightarrow \boxed{c = 2^{\aleph_0}}$$

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Ques) Let X be any set & $C(X)$ be the family of characteristic fns of X i.e " " " functions $f: X \longrightarrow \{0,1\}$ then prove that $P(X) \cong C(X)$

Proof:- Let A be any subset of X
Hence $A \in P(X)$
Let $f: P(X) \longrightarrow C(X)$

$$f(A) = g_A$$

i.e f maps each subset A of X into the characteristic function g_A of A relative to X. then it's obvious that f is 1-1 & onto.
Hence $P(X) \cong C(X)$

☆ If $E(X,Y)$ be the set of all the functions from $X \rightarrow Y$ i.e $E(X,Y)$ contains those $f: X \rightarrow Y$ then cardinality of $E(X,Y)$ $[|E(X,Y)|] = |Y|^{|X|}$

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Cardinal Arithmetic
" Addition & Multiplication
Addition & multiplication of counting No. IN are sometimes treated from the point of few set theory

Def:- Let α & β be cardinal no. of set A & B $\{A\}$ & $\{B\}$ then the sum of α & β is denoted by & defined by

$\alpha = |A|$

$$\alpha + \beta = |(A \cup B)|$$

$\beta = |B|$

$A \cap B = \Phi$

$f(m) = n + m$ (n fixed)

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Ex:- Let $n \to$ finite cardinal no. of A
& $\aleph_0 \to$ transfinite " " of B
then $n + \aleph_0 = |\{1, 2, 3, \ldots, n\} \cup \{n+1, n+2, \ldots\}|$

$$\boxed{\begin{aligned} n + \aleph_0 &= |\mathbb{N}| \\ &= \aleph_0 \end{aligned}}$$ finite + infinite = infinite

Ex:- $\aleph_0 + \aleph_0 = |\{2, 4, 6, \ldots\} \cup \{1, 3, 5, 7, \ldots\}|$
$= |\mathbb{N}|$
$= \aleph_0$

Ex:- To prove $c + c = c$ $\cong [0,1]$ $\cong [0,1]$
$= |[2,3] \cup [3,4]|$
$= |[2,4]| \longrightarrow \cong [0,1]$
$= c$

Multiplication:-
Let α & β be cardinal no. of set A & set B then product of α & β is denoted & defined by
$$\alpha\beta = |A \times B|$$

Ex:- $|A| = m$
$|B| = n$ {finite cardinal no.}
$mn = |A \times B|$

Ex:- since $\overbrace{\mathbb{N} \times \mathbb{N}}^{\cong \mathbb{N}}$ (denumerable set)
$\aleph_0 \aleph_0 = |\mathbb{N} \times \mathbb{N}| = \aleph_0$

Ex:- The cartesian plane $\mathbb{R}^2$ has same cardinality

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as $\mathbb{R}$.
$$|\mathbb{R}\times\mathbb{R}| = |\mathbb{R}|$$
$$\Rightarrow \quad c \cdot c = c$$

Theorem :- Addition & Multiplication of cardinal no. satisfy the properties in table A.

table "A"

Cardinal No.	Sets
i) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$	i) $(A\cup B)\cup C = A\cup(B\cup C)$
ii) $\alpha+\beta = \beta+\alpha$	ii) $A\cup B = B\cup A$
iii) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$	iii) $(A\times B)\times C \approx A\times(B\times C)$
iv) $\alpha\beta = \beta\alpha$	iv) $A\times B \approx B\times A$
v) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$	v) $A\times(B\cup C) = (A\times B)\cup(A\times$
vi) If $\alpha\le\beta$ then $\alpha+\gamma\le\beta+\gamma$	vi) If $A\subseteq B$ then $(A\cup C)\subseteq(B\cup C)$
vii) If $\alpha\le\beta$ then $\alpha\gamma\le\beta\gamma$	vii) If $A\subseteq B$ then $(A\times C)\subseteq(B\times C)$

Theorem:- Let α & β be the non zero cardinal no. & β is infinite, their sum & product is simply the larger of 2.

$\alpha\le\beta$ then $\quad \alpha+\beta = \alpha\cdot\beta = \beta$

$c = 2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$

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Note:- If A & B are the sets then A^B denotes set of all fn $B \to A$ i.e $f: B \to A$

$f: \mathbb{N} \to \{0,1\}$

$|\{0,1\}|^{|\mathbb{N}|} = 2^{\aleph_0}$

def:- Let α & β be the cardinal no. of the sets A & B then α^β denotes $|A^B|$

Theorem:- Let α, β, γ be the cardinal no. then

i) $(\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$

ii) $\alpha^\beta \alpha^\gamma = \alpha^{\beta+\gamma}$

iii) $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$

iv) If $\alpha \leq \beta$ then $\alpha^\gamma \leq \beta^\gamma$

Ex:- a) $c^{\aleph_0} = c$

LHS

$= (2^{\aleph_0})^{\aleph_0}$

$= 2^{\aleph_0 \aleph_0}$

$= 2^{\aleph_0}$

$= c$

$2^{2^{\aleph_0}}$

b) $c^c = 2^c$

LHS

$(2^{\aleph_0})^c$

$= 2^{\aleph_0 \cdot c}$ $c > \aleph_0$

$= 2^c$

Prove that $\aleph_0 c = c$

consider $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ &

$|B| = \aleph_0$ $\qquad \aleph_0 = c$

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$3^A \rightarrow$ set of all fn $f: A \rightarrow B$

$A = [0, 1)$.

Furthermore let $f: \mathbb{Z} \times A = \mathbb{R}$

defined by $f(n, a) = n + a$

In other words $f(\{n\} \times [0,1))$ is map onto $[n, n+1)$

Then f is one-one correspondance b/w $\mathbb{Z} \times A$ & $\mathbb{R}$. $\because$ $|\mathbb{Z}| = \aleph_0$ & $|A| = c = |\mathbb{R}|$

Hence we have $\aleph_0 \cdot c = |\mathbb{Z} \times A| = |\mathbb{R}| = c$

Prove that α be any infinite cardinal no. then
$\aleph_0 + \alpha = \alpha$

we have shown that $\aleph_0 + \aleph_0 = \aleph_0$

suppose $\{A\}$ is uncountable then $\alpha = |A|$

Now $A \setminus B \simeq A$ where B is denumerable subset of A. Recall $A = \cancel{A \setminus} (A \setminus B) \cup B$ & the union is disjoint.

Hence $\alpha = |A| = |(A \setminus B) \cup B| = |A \setminus B| + |B|$

$= \alpha + \aleph_0$

$\Rightarrow \alpha = \alpha + \aleph_0$

Result: $\{P\}$ of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ with integral coeff. i.e a_i's are integers is denumerable. (set $\mathbb{Z}$ is denumerable?)

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Each pair of non negative integer (n, m), le $P(n, m)$ be the set of polynomials in I of degree m. In which $|a_0| + |a_1| + \cdots + |a$

$=$

Of ... uncountable

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degree of poly

where $a_0, a_1, a_2, \ldots, a_m$ are coeff. of polyⁿ.
Note that $P(n,m)$ is finite.
Therefore $P = \bigcup (P(n,m))$ s.t $n,m \in (\mathbb{N} \times \mathbb{N})$
is countable. Since it's a countable
family of " sets. But P isn't finite.
Hence P is denumerable.

Prove that $\mathbb{Q}^c$ is uncountable.

$$\mathbb{Q} \cup \mathbb{Q}^c = \mathbb{R}$$

WKT $\mathbb{R}$ is uncountable, $\mathbb{Q}$ is countable
let $\mathbb{Q}^c$ is countable
Union of countable set is countable.
There is a contradiction.
$\therefore$ $\mathbb{Q}^c$ can't be countable.

Theorem:- Set of all algebraic number is denumerable.

Set $E = \{ p_1(x) = 0, p_2(x) = 0, \ldots \}$

$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$
where $a_i \in \mathbb{Q}$
Now define $A_k = \{ x : x \text{ is a sol}^n \text{ of } p_k(x) = 0 \}$

Since a polyⁿ of n degree can have atmost n ~~loops~~ roots. Hence each A_k is finite.
$\therefore$ $A = \bigcup \{ A_k, k \in \mathbb{N} \}$
A is countable family of countable set.

Accordingly X is countable but not finite.
Hence X is denumerable.

Theorem ~~Now~~ prove that set of all transcedental no. is uncountable.

Pf:- let transcedental no. be countable
& $\mathbb{R}$ is uncountable
Union of algebraic no. & $\mathbb{R}$

Algebraic no. $\cup$
Real no. = Real no.

Partial order set
Antisymmetric Relation :- A relation $\mathbb{R}$ on a set S is said to antisymmetric if aRb & bRa (b Related to a) then $a=b$

A relation $\mathbb{R}$ on a set S is called partial ordering of S or a partial order on S if it has the following properties. aRa $\forall$ $a \in S$

$[S; R]$

only one should be include o/w $a=b$

1) Reflexive aRa, $\forall a \in S$
2) Antisymmetric aRb & bRa, then $a=b$
3) R is transitive aRb, bRc then aRc

A set S together with a partial ordering R is called a partially ordered set or poset

Ex:- If set S is any collection of sets with the relation $R = \subseteq$ (~~subset~~ set inclusion)
$(R = \subseteq)$

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then $(S, \subseteq)$ is partially ordered set.

Pf i) Reflexive → For $A \in S$, $A \subseteq A \Rightarrow ARA \Rightarrow R$ is reflexive

ii) Antisymmetric → If $A, B \in S$, $A \subseteq B$ & $B \subseteq A \Rightarrow A = B$
$\Rightarrow R$ is antisymmetric

iii) Transitive If $A, B, C \in S$
$A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$

Ex:- $(\mathbb{R}, \leq)$ forms poset ?

i) $a \leq a$

ii) $a \leq b$ & $b \leq a \Rightarrow a = b$

iii) $a \leq b$, $b \leq c \Rightarrow a \leq c$

The relation $a|b$ is partial ordering of the set of all +ve integers. However $a|b$ is not partial ordering of the set of all integers because $a|b$ & $b|a$ doesn't imply $a = b$

Ex:- $3|-3$ & $-3|3$ but $3 \neq -3$

① aRa because $a|a$

② If $a|b$ & $b|a$ when $a = b$ (not possible for -ve integers)

$3|12$, $12 \nmid 3$ Anti

Relation of divisibility is a POSET (not in -ve Z)