# Module1 : Lecture 1
# BASIC CONCEPTS AND PROPERTIES OF FLUIDS

## BASIC CONCEPTS

- *Mechanics* is the oldest physical science that deals with both stationery and moving boundaries under the influence of forces. The branch of the mechanics that deals with bodies at rest is called *statics* while the branch that deals with bodies in motion is called *dynamics*.

- *Fluid Mechanics* is the science that deals with behavior of fluids at rest (fluid statics) or in motion (fluid dynamics) and the interaction of fluids with solids or other fluids at the boundaries.

- A substance in liquid / gas phase is referred as 'fluid'. Distinction between a solid & a fluid is made on the basis of substance's ability to resist an applied shear (tangential) stress that tends to change its shape. A solid can resist an applied shear by deforming its shape whereas a fluid deforms continuously under the influence of shear stress, no matter how small is its shape. In solids, stress is proportional to strain, but in fluids, stress is proportional to 'strain rate.'
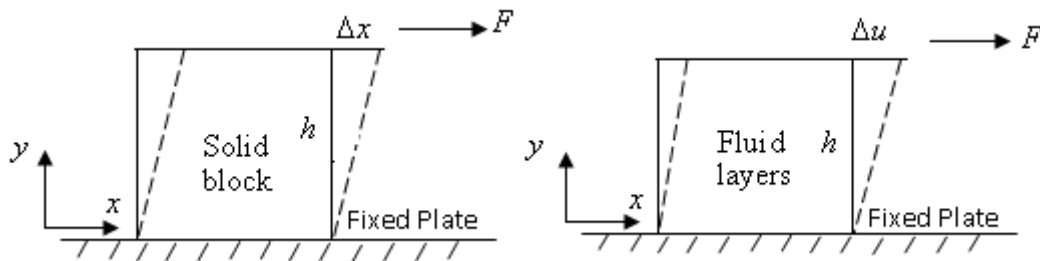


Fig. 1.1.1: Illustration of solid and fluid deformation.

Referring to Fig. 1.1.1, the shear modulus of solid $(S)$ and coefficient of viscosity $(\mu)$ for fluid can defined in the following manner;

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{(F/A)}{(\Delta x/h)}; \quad \mu = \frac{\text{Shear stress}}{\text{Shear strain rate}} = \frac{(F/A)}{(\Delta u/h)} \tag{1.1.1}$$

Here, the shear force $(F)$ is acting on the certain cross-sectional area $(A)$, $h$ is the height of the solid block / height between two adjacent layer of the fluid element, $\Delta x$ is the elongation of the solid block and $\Delta u$ is the velocity gradient between two adjacent layers of the fluid.

- Although liquids and gases share some common characteristics, they have many distinctive characteristics on their own. It is easy to compress a gas whereas liquids are incompressible. A given mass of the liquid occupies a fixed volume, irrespective of the size and shape of the container. A gas has no fixed volume and will expand continuously unless restrained by the containing vessel. For liquids a free surface is formed in the volume of the container is greater than that of the liquid. A gas will completely fill any vessel in which it is placed and therefore, does not have a free surface.

**Dimension and Unit**

A *dimension* is the measure by which a physical variable is expressed quantitatively and the *unit* is a particular way of attaching a number to the quantities of dimension. All the properties of fluid are assigned with certain unit and dimension. Some basic dimensions such as mass (M), length (L), time (T) and temperature ($\theta$) are selected as *Primary/Fundamental* dimensions/unit. While others such as velocity, volume is expressed in terms of primary dimensions and is called as *secondary/derived* dimensions/unit. In this particular course, SI (Standard International) system of units and dimension will be followed to express the properties of fluid.

**Fluid as Continuum**

Fluids are aggregations of molecules; widely spaced for a gas and closely spaced for liquids. Distance between the molecules is very large compared to the molecular diameter. The number of molecules involved is immense and the separation between them is normally negligible. Under these conditions, fluid can be treated as *continuum* and *the properties at any point can be treated as bulk behavior of the fluids*.

For the continuum model to be valid, the smallest sample of matter of practical interest must contain a large number of molecules so that meaningful averages can be calculated. In the case of air at sea-level conditions, a volume of $10^{-9}$mm$^3$ contains $3\times10^7$ molecules. In engineering sense, this volume is quite small, so the continuum hypothesis is valid.

In certain cases, such as, very-high-altitude flight, the molecular spacing becomes so large that a small volume contains only few molecules and the continuum model fails. For all situations in these lectures, the continuum model will be valid.

**Properties of Fluid**

Any characteristic of a system is called *property*. It may either be *intensive* (mass independent) or *extensive* (that depends on size of system). The state of a system is described by its properties. The number of properties required to fix the state of the system is given by *state postulates*. Most common properties of the fluid are:

**1. Pressure** $(p)$: It is the normal force exerted by a fluid per unit area. More details will be available in the subsequent section (Lecture 02). In SI system the unit and dimension of pressure can be written as, N/m$^2$ and $ML^{-1}T^{-2}$, respectively.

**2. Density**: The density of a substance is the quantity of matter contained in unit volume of the substance. It is expressed in three different ways; mass density $\left( \rho = \dfrac{\text{mass}}{\text{volume}} \right)$, specific weight $(\rho g)$ and relative density/specific gravity $\left( SG = \dfrac{\rho}{\rho_{\text{water}}} \right)$. The units and dimensions are given as,

      For mass density;     Dimension: $ML^{-3}$        Unit: kg/m$^3$

      For specific weight;   Dimension: $ML^{-2}T^{-2}$     Unit: N/m$^3$

The standard values for density of water and air are given as 1000kg/m$^3$ and 1.2 kg/m$^3$, respectively. Many a times the reciprocal of mass density is called as specific volume $(\forall)$.

**3. Temperature** $(T)$: It is the measure of hotness and coldness of a system. In thermodynamic sense, it is the measure of internal energy of a system. Many a times, the temperature is expressed in centigrade scale (°C) where the freezing and boiling point of water is taken as 0°C and 100°C, respectively. In SI system, the temperature is expressed in terms of absolute value in Kelvin scale (K = °C+ 273).

**4. Viscosity** $(\mu)$: When two solid bodies in contact, move relative to each other, a friction force develops at the contact surface in the direction opposite to motion. The situation is similar when a fluid moves relative to a solid or when two fluids move relative to each other. The property that represents the internal resistance of a fluid to motion (i.e. *fluidity*) is called as *viscosity*. The fluids for which the rate of deformation is proportional to the shear stress are called Newtonian fluids and the linear relationship for a one-dimensional system is shown in Fig. 1.1.2. The shear stress $(\tau)$ is then expressed as,

$$\tau = \mu \frac{du}{dy} \tag{1.1.2}$$

where, $\frac{du}{dy}$ is the shear strain rate and $\mu$ is the dynamic (or absolute) viscosity of the fluid.

The dynamic viscosity has the dimension $ML^{-1}T^{-1}$ and the unit of kg/m.s (or, N.s/m$^2$ or Pa.s) . A common unit of dynamic viscosity is *poise* which is equivalent to 0.1 Pa.s. Many a times, the ratio of dynamic viscosity to density appears frequently and this ratio is given by the name kinematic viscosity $\left( v = \frac{\mu}{\rho} \right)$. It has got the dimension of $L^2T^{-1}$ and unit of *stoke* (1 stoke = 0.0001 m$^2$/s). Typical values of kinematic viscosity of air and water at atmospheric temperature are 1.46 x 10$^{-5}$ m$^2$/s and 1.14 x 10$^{-6}$ m$^2$/s, respectively.
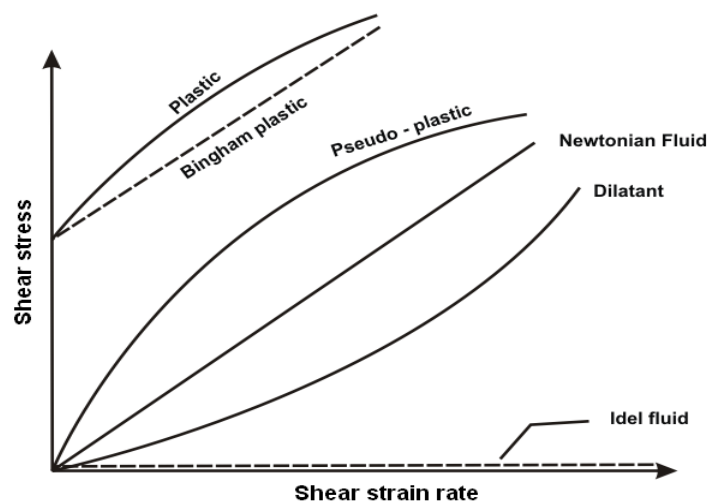


**Fig. 1.1.2: Variation of shear stress with rate of deformation.**

In general, the viscosity of a fluid mainly depends on temperature. For liquids, the viscosity decreases with temperature and for gases, it increases with temperature. Sutherland's correlation is used to determine viscosity of gases as a function of temperature.

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \left(\frac{T_0 + S}{T + S}\right) \tag{1.1.3}$$

For air, the reference value of viscosity $\mu_0 = 1.789 \times 10^{-5}\,\text{kg/m.s}$ at $T_0 = 288\,\text{K}$ and $S = 110\,\text{K}$. In the case of liquids, the viscosity is approximated as below;

$$\ln\left(\frac{\mu}{\mu_0}\right) = a + b\left(\frac{T_0}{T}\right) + c\left(\frac{T_0}{T}\right)^2 \tag{1.1.4}$$

For water at $T_0 = 273\ \text{K}$, $\mu_0 = 0.001792\ \text{kg/m.s}$, $a = -1.94, b = -4.8, c = 6.74$.

**5. Thermal Conductivity** $(k)$: It relates the rate of heat flow per unit area $(\dot{q})$ to the temperature gradient $\left(\dfrac{dT}{dx}\right)$ and is governed by Fourier Law of heat conduction i.e.

$$\dot{q} = -k\frac{dT}{dx} \tag{1.1.5}$$

In SI system the unit and dimension of pressure can be written as, W/m.K and $\text{MLT}^{-3}\theta^{-1}$, respectively. Thermal conductivity varies with temperature for liquids as well as gases in the same manner as that of viscosity. The reference value of thermal conductivity $(k_0)$ for water and air at reference temperature is taken as, 0.6 W/m.K and 0.025 W/m.K, respectively.

**6. Coefficient of compressibility/Bulk modulus** $(E_v)$: It is the property of that fluid that represents the variation of density with pressure at constant temperature. Mathematically, it is represented as,

$$E_v = -\mathrm{\forall}\left(\frac{\partial p}{\partial \mathrm{\forall}}\right)_T = \rho\left(\frac{\partial \rho}{\partial T}\right)_T \tag{1.1.6}$$

In terms of finite changes, it is approximated as,

$$E_v = \frac{\left(\Delta\Psi/\Psi\right)}{\Delta T} = -\frac{\left(\Delta\rho/\rho\right)}{\Delta T} \qquad (1.1.7)$$

It can be shown easily that $E_v$ for an ideal gas at a temperature $p$ is equal to its absolute pressure (N/m$^{-2}$).

**7. Coefficient of volume expansion** $\left(\beta\right)$: It is the property of that fluid that represents the variation of density with temperature at constant pressure. Mathematically, it is represented as,

$$\beta = \frac{1}{\Psi}\left(\frac{\partial\Psi}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \qquad (1.1.8)$$

In terms of finite changes, it is approximated as,

$$\beta = \frac{\left(\Delta\Psi/\Psi\right)}{\Delta T} = -\frac{\left(\Delta\rho/\rho\right)}{\Delta T} \qquad (1.1.9)$$

It can be shown easily that $E_v$ for an ideal gas at a temperature $T$ is equivalent to inverse of the absolute temperature.

**8. Specific heats**: It is the amount of energy required for a unit mass of a fluid for unit rise in temperature. Since the pressure, temperature and density of a gas are interrelated, the amount of heat required to raise the temperature from $T_1$ to $T_2$ depends on whether the gas is allowed to expand during the process so that the energy supplied is used in doing the work instead of raising the temperature. For a given gas, two specific heats are defined corresponding to the two extreme conditions of constant volume and constant pressure.

    (a) Specific heat at constant volume $\left(c_v\right)$

    (b) Specific heat at constant pressure $\left(c_p\right)$

The following relation holds good for the specific heat at constant volume and constant pressure. For air ; $c_p = 1.005$ KJ/kg.K    $c_v = 0.718$ KJ/kg. K

$$c_p - c_v = R; \quad c_p = \frac{\gamma R}{\gamma - 1}; \quad c_v = \frac{R}{\gamma - 1} \tag{1.1.10}$$

**9. Speed of sound** $(c)$: An important consequence of compressibility of the fluid is that the disturbances introduced at some point in the fluid propagate at finite velocity. The velocity at which these disturbances propagate is known as "acoustic velocity/speed of sound". Mathematically, it is represented as below;

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{E_v}{\rho}} \tag{1.1.11}$$

In an isothermal process,

$$E_v = p \implies c = \sqrt{\frac{p}{\rho}}$$
$$c = \sqrt{RT} \;\text{ (for an ideal gas medium)} \tag{1.1.12}$$

In isentropic process,

$$E_v = \gamma p \implies c = \sqrt{\frac{\gamma p}{\rho}}$$
$$c = \sqrt{\gamma RT} \;\text{ (for an ideal gas medium)} \tag{1.1.13}$$

**10. Vapour pressure** $(p_v)$: It is defined as the pressure exerted by its vapour in phase equilibrium with its liquid at a given temperature. For a pure substance, it is same as the saturation pressure. In a fluid motion, if the pressure at some location is lower than the vapour pressure, bubbles start forming. This phenomenon is called as cavitation because they form cavities in the liquid.

**11. Surface Tension** $(\sigma)$: When a liquid and gas or two immiscible liquids are in contact, an unbalanced force is developed at the interface stretched over the entire fluid mass. The intensity of molecular attraction per unit length along any line in the surface is called as surface tension. For example, in a spherical liquid droplet of radius $(r)$, the pressure difference $(\Delta p)$ between the inside and outside surface of the droplet is given by,

$$\Delta p = \frac{2\sigma}{r} \qquad (1.1.14)$$

In SI system the unit and dimension of pressure can be written as, N/m and $MT^{-2}$, respectively.

**State Relations for Gases and Liquids**

All gases at high temperatures and low pressures are in good agreements with 'perfect gas law' given by,

$$p = \rho R T = \rho \left(\frac{\bar{R}}{M}\right) T \qquad (1.1.15)$$

where, $R$ is the characteristic gas constant, $\bar{R}$ is the universal gas constant and $M$ is the molecular weight.

Liquids are nearly incompressible and have a single reasonable constant specific heat. Density of a liquid decreases slightly with temperature and increases moderately with pressure. Neglecting the temperature effect, an empirical pressure- density relation is expressed as,

$$\frac{p}{p_a} = (B+1)\left(\frac{\rho}{\rho_a}\right)^n - B \qquad (1.1.16)$$

Here, $B$ and $n$ are the non-dimensional parameters that depend on the fluid type and vary slightly with the temperature. For water at 1 atm, the density is 1000 kg/m$^3$ and the constants are taken as, $B = 3000$ and $n = 7$

**Classifications of Fluid Flows**

Some of the general categories of fluid flow problems are as follows;

1. Viscous and  Inviscid flow: The fluid flow in which frictional effects become signification, are treated as viscous flow. When two fluid layers move relatively to each other, frictional force develops between them which is quantified by the fluid property 'viscosity'. Boundary layer flows are the example viscous flow. Neglecting the viscous terms in the governing equation, the flow can be treated as inviscid flow.

2. Internal and External flow:  The flow of an unbounded fluid over a surface is treated as 'external flow' and if the fluid is completely bounded by the surface, then it is called as 'internal flow'. For example, flow over a flat plate is considered as external flow and flow through a pipe/duct is internal flow. However, in special cases, if the duct is partially filled and there is free surface, then it is called as open channel flow. Internal flows are dominated by viscosity whereas the viscous effects are limited to boundary layers in the solid surface for external flows.

3. Compressible and Incompressible flow:   The flow is said to be 'incompressible' if the density remains nearly constant throughout. When the density variation during a flow is more than 5% then it is treated as 'compressible'. This corresponds to a flow Mach number of 0.3 at room temperature.

4. Laminar and Turbulent flow: The highly ordered fluid motion characterized by smooth layers of fluid is called 'Laminar Flow', e.g. flow of highly viscous fluids at low velocities. The fluid motion that typically occurs at high velocities is characterized by velocity fluctuations are called as 'turbulent.' The flow that alternates between being laminar & turbulent is called 'transitional'. The dimensionless number i.e. Reynolds number is the key parameter that determines whether the flow is laminar or turbulent.

5. Steady and Unsteady flow:  When there is no change in fluid property at point with time, then it implies as steady flow. However, the fluid property at a point can also vary with time which means the flow is unsteady/transient. The term 'periodic' refers to the kind of unsteady flows in which the flow oscillates about a steady mean.

5. Natural and Forced flow:  In a forced flow, the fluid is forced to flow over a surface by external means such as a pump or a fan. In other case (natural flow), density

difference is the driving factor of the fluid flow. Here, the buoyancy plays an important role. For example, a warmer fluid rises in a container due to density difference.

6. One/Two/Three dimensional flow:  A flow field is best characterized by the velocity distribution, and thus can be treated as one/two/three dimensional flow if velocity varies in the respective directions.

# Module 1 : Lecture 2
# REVIEW OF FLUID STATICS

## FLUID STATICS

Many fluid problems do not involve motion rather concerned with the pressure distribution in a static fluid. When the fluid velocity is *zero*, known as *hydrostatic condition*, the pressure variation is due to *weight* of the fluid. The important areas of fluid statics include;

- Pressure distribution in atmospheres and oceans

- Design of manometer pressure instruments

- Forces on submerged flat and curved surfaces

- Buoyancy on a submerged body

- Behavior of floating bodies

### Hydrostatic pressure for fluids

- The term 'pressure' is used to indicate the normal force per unit area at a point acting on a given plane within the fluid mass of interest. It is governed by *Pascal's law which states that the pressure at a point in a fluid at rest or in motion is independent of direction as long as there is no shearing stress present.*

- The net pressure force acting an fluid element per unit volume $\left( \vec{f}_{\text{pressure}} \right)$ is given by,

$$\vec{f}_{\text{pressure}} = -\nabla p \qquad (1.2.1)$$

   It is not the pressure but the pressure gradient that causes the net force.

- In engineering applications, the pressure is measured either as absolute numbers (called *absolute pressure*) or relative to atmospheric value (*gauge/vacuum pressure*). If the atmospheric pressure is given as $p_a$, then referring to Fig. 1.2.1,

   - Pressure at 'A' can be measured as gauge pressure $\left( p_{\text{gauge}} \right)$ or absolute pressure $\left( p_1 \right)$ i.e.

$$p_1 = p_a + \Delta p_1; \quad p_{\text{gauge}} = \Delta p_1 = p_1 - p_a \qquad (1.2.2)$$

- Pressure at 'B' can be measured as vacuum pressure $(p_{vacuum})$ or absolute pressure $(p_2)$ i.e.

$$p_2 = p_a - \Delta p_2; \quad p_{vacuum} = \Delta p_2 = p_a - p_2 \qquad (1.2.3)$$

The value of local atmosphere is given by,

$$1\,\text{atm} = 1.01325\,\text{bar} = 760\,\text{torr}\,(\text{mm of Hg})$$
$$= 101.325\,\text{kPa} = 101.325\,\text{kN}/\text{m}^2$$
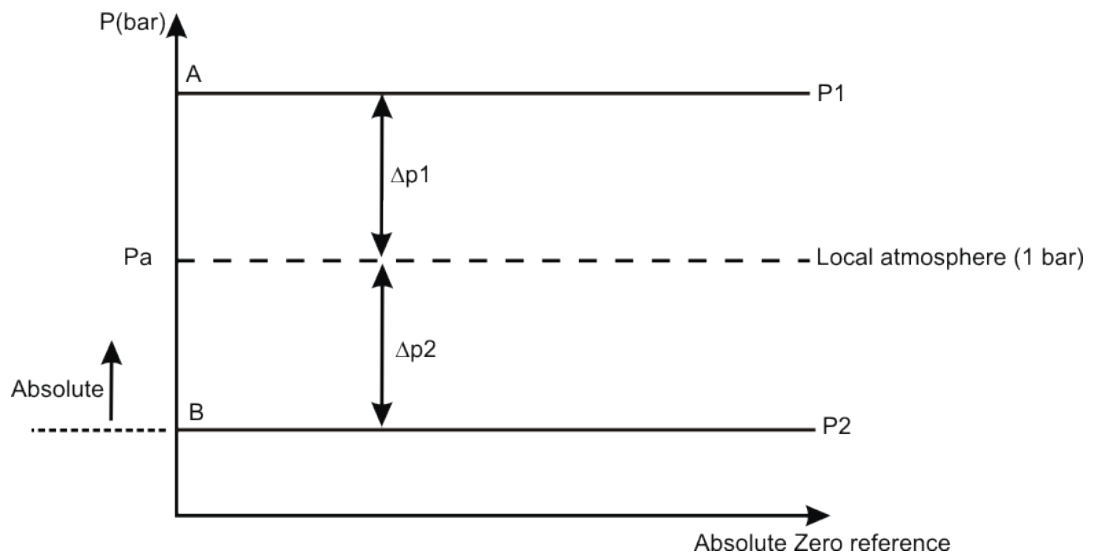$$= 14.7\,\text{psi}\,(\text{pounds per square inch})$$



Fig. 1.2.1: Illustration of gauge and vacuum pressure.

- The pressure gradient is a surface force that acts on the sides of a fluid element. Also, if the fluid element is in motion, it will have surface forces due to viscous stresses. There may be body force due to gravitational potential, acting on the entire mass of the element. By, Newton's second law, the sum of these forces per unit volume $\left(\vec{f}\right)$ equals to the density $(\rho)$ times (mass per unit volume) the acceleration $(\vec{a})$ of the fluid element, i.e.

$$\sum \vec{f} = \vec{f}_{pressure} + \vec{f}_{gravity} + \vec{f}_{viscous} = -\nabla p + \rho \vec{g} + \vec{f}_{viscous}$$
$$\Rightarrow \rho \vec{a} = -\nabla p + \rho \vec{g} + \vec{f}_{viscous} \qquad (1.2.4)$$

This is the general equilibrium equation for a fluid element. Assuming the inviscid assumption $\left( \vec{f}_{viscous} \right)$, when the fluid is at rest or at constant velocity $\left( \vec{a} = 0 \right)$, the pressure distribution reduces to,

$$\nabla p = \rho \vec{g} \qquad (1.2.5)$$

This is hydrostatic pressure distribution and is correct for all fluids at rest, regardless of their viscosity.

- In Eq. (1.2.5), $\nabla p$ expresses the magnitude and direction of the maximum spatial rate of increase of the scalar property $p$ and is perpendicular everywhere to the surface. The hydrostatic equilibrium will align constant-pressure surfaces everywhere normal to gravity vector. In the customary coordinate system, the direction of $z$ is opposite to the direction of gravity (down), the Eq. (1.2.5) can be written in scalar form as,

$$\frac{\partial p}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = 0; \quad \frac{\partial p}{\partial z} = \frac{dp}{dz} - \rho g = -\gamma \qquad (1.2.6)$$

where $\gamma$ is the specific weight of the fluid. Integrating the above equation,

$$p_2 - p_1 = -\int_1^2 \gamma \, dz \qquad (1.2.7)$$

- Thus, the important conclusion about hydrostatic condition is as follows: "Pressure in a continuously distributed uniform static fluid varies only with vertical distance and is independent of the shape of the container. The pressure is the same at all point on a given horizontal plane in the fluid. The pressure increases/decreases with depth of the fluid. The hydrostatic pressure distribution in oceans and atmospheres is shown in Fig. 1.2.2. Considering the sea-level for the reference $\left( z = 0;\ p = p_a \right)$, the pressure decreases with altitude $\left( b \right)$ where the fluid medium is air $\left( \gamma_{air} \right)$. For an ocean, when the depth $\left( h \right)$ is increased (vertically downward) the pressure gets increased for water $\left( \gamma_{water} \right)$.
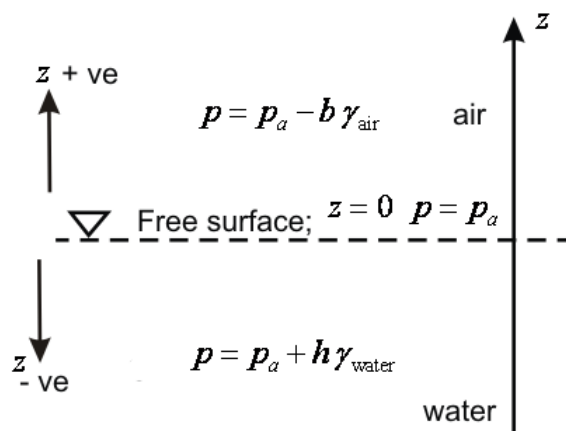


**Fig. 1.2.2: Hydrostatic pressure distribution in oceans and atmosphere.**

- Liquids are generally incompressible so that the density variations are neglected in hydrostatics. So, Eq. (1.2.7) reduces to the following form;

$$z_1 - z_2 = \frac{p_2}{\gamma} - \frac{p_1}{\gamma} \tag{1.2.8}$$

where the term $\dfrac{p}{\gamma}$ is called as *pressure head* of the fluid.

- Gases are compressible with density proportional to pressure. Apply perfect gas law, in Eq. (1.2.6).

$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT} g \tag{1.2.9}$$

where, $R$ is the characteristics gas constant and $T$ is the absolute temperature. Integration of above equation yields,

$$\ln\left(\frac{p_2}{p_1}\right) = -\frac{g}{R}\int_1^2 \frac{dz}{T} \tag{1.2.10}$$

This equation needs the assumption of temperature variation $T(z)$ e.g. in an isothermal atmosphere, where $T = T_0$,

$$p_2 = p_1 \exp\left[\frac{g(z_2 - z_1)}{RT_0}\right] \tag{1.2.11}$$

**Manometer pressure instruments**

A change in elevation of a liquid is equivalent to the change in pressure (Eq. 1.2.8). Thus a static column of one/more liquids/gases can be used to measure the pressure difference between two points. Such a device is called a *manometer*. The simplest type of U-tube manometer is shown in Fig. 1.2.3. In the case of open U-tube manometer (Fig. 1.2.3-a), one end is connected to a reservoir 'A' and other end is open to atmosphere. The hydrostatic formula applicable for this case is,

$$p_A + \gamma_1(z_A - z_1) - \gamma_2(z_1 - z_2) = p_2 = p_{\text{atm}} \tag{1.2.12}$$

In case of multi-fluid manometer as shown in Fig. 1.2.3(b), both ends of the tube are connected to the reservoirs 'A' and 'B'. The pressure difference can then be computed as,

$$
\begin{aligned}
p_A - p_B &= (p_A - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_B) \\
&= -\gamma_1(z_A - z_1) + \gamma_2(z_2 - z_1) - \gamma_3(z_2 - z_3) + \gamma_4(z_B - z_3)
\end{aligned}
\tag{1.2.13}
$$



(a) Simple open manometer    (b) Multi-fluid U-tube manometer
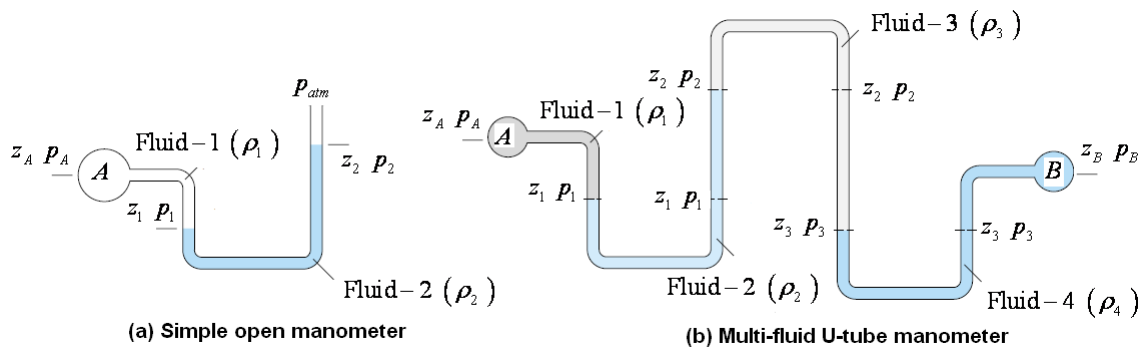
Fig. 1.2.3: U-tube manometer for determination of hydrostatic pressure.

**Hydrostatic force on a plane surface**

In the design of storage tanks, ships, dams and many hydraulic static structures, the determination of hydrostatic force is very important. When a surface is submerged in a fluid, forces develop perpendicular to the surface since there is no shearing stress. Also, the pressure will vary linearly with depth if the fluid is incompressible. As shown in Fig. 1.2.4(a), the magnitude of the resultant force at the bottom of a liquid filled tank is given by,

$$F_R = pA = (\gamma h) A \qquad (1.2.14)$$

Since the atmospheric pressure acts on both sides of the bottom surface, the resultant force is purely due to the weight of the liquid in the tank and acts at the centroid of the area occupied by the liquid. In contrast, when the pressure is not uniformly distributed (Fig. 1.2.4-b), i.e. the pressure varies linearly in the side wall. In such cases, the magnitude and direction of resultant force must be determined from the general expressions given below.
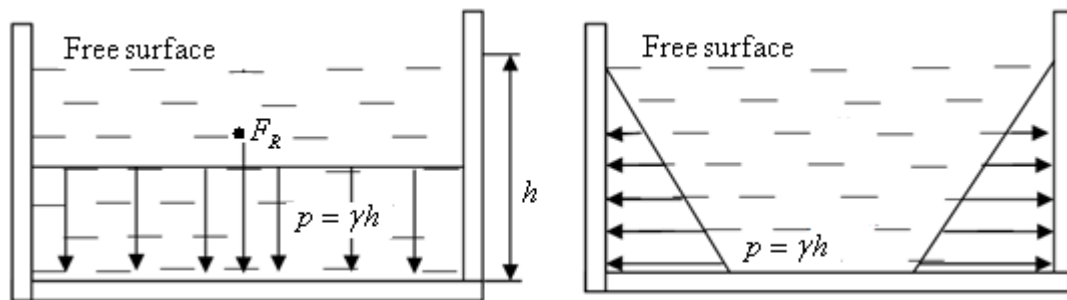


**Fig. 1.2.4: Pressure distribution and resultant force in an open tank:**

**(a) bottom portion of the tank; (b) side of the tank.**

In the most general case, we consider an inclined submerged plane surface as shown in Fig. 1.2.5. Let the plane in which the surface lies, intersect the free surface at 'O' and make an angle $\theta$ with the surface. The $x - y$ coordinates are defined such that 'O' is the origin and all other geometric parameters are given in the Fig. 1.2.5. The area can have any arbitrary shape and it is desired to determine the direction, location and magnitude of the resultant force acting on one side of this area due to the liquid in contact with this area. At any given depth $(h)$, the force acting on the differential area $dA$ is $dF = (\gamma h) dA$ and acts perpendicular to the surface. Integrating this expression, the resultant force may be found from the following analysis;

$$F_R = \int_A (\gamma h) dA = \int_A (\gamma y \sin \theta) dA = \gamma \sin \theta \int_A y \, dA = \gamma A \, y_c \sin \theta = \gamma h_c A \qquad (1.2.15)$$

Here, $\int\limits_{A} y\, dA = y_c A$ is the first moment of inertia with respect to $x$-axis, $y_c$ is the $y$-coordinate of the centroid area 'A' measured from $x$-axis which passes through 'O' and $h_c$ is the vertical distance from the fluid surface to the centroid of the area. Thus, it is clear from the Eq. (1.2.15) that the magnitude of the force is independent of the orientation of the surface and only depends on the specific weight of the fluid, total area and depth of the centroid of the area below the free surface.
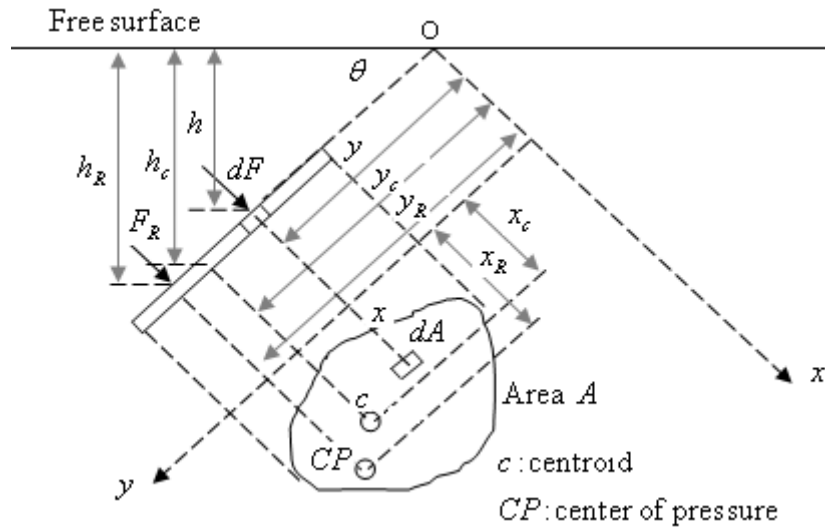


**Fig. 1.2.5: Notations for hydrostatic force on any arbitrary inclined plane surface.**

It is not always necessary that the resultant force will act at the centroid of the area. In order to calculate the location of this resultant force, let us take the moment of the force and it must be equal to the moment of the distributed pressure force. Considering the summation of moments about $x$-axis,

$$F_R y_R = \int\limits_{A} y\, dF = \int\limits_{A} (\gamma \sin\theta)\, y^2 dA = \gamma \sin\theta \int\limits_{A} y^2\, dA$$

$$\Rightarrow (\gamma A\, y_c \sin\theta)\, y_R = \gamma \sin\theta \int\limits_{A} y^2\, dA \qquad (1.2.16)$$

$$\Rightarrow y_R = \frac{\int\limits_{A} y^2\, dA}{y_c A} = \frac{I_x}{y_c A}$$

Here, $I_x$ is the 'moment of inertia' with respect to an axis formed by the intersection of the plane containing the surface and free surface ($x$-axis). Using parallel axis theorem, one may express $I_x$ in terms of second moment of inertia with respect to centroid $(I_{xc})$ and parallel to x- axis i.e. $I_x = I_{xc} + Ay_c^2$ . Thus, Eq. (1.2.16) becomes,

$$y_R = \frac{I_{xc}}{y_c A} + y_c \tag{1.2.17}$$

In a similar manner, the $x$-coordinate of the resultant force can be determined by summing the moments about $y$-axis.

$$F_R x_R = \int_A x\,dF = \int_A (\gamma \sin\theta)\,xy\,dA = \gamma \sin\theta \int_A y^2\,dA$$

$$\Rightarrow (\gamma A\,y_c \sin\theta)\,x_R = \gamma \sin\theta \int_A xy\,dA \tag{1.2.18}$$

$$\Rightarrow x_R = \frac{\int_A xy\,dA}{y_c A} = \frac{I_{xy}}{y_c A} = \frac{I_{xyc} + A x_c y_c}{y_c A} = \frac{I_{xyc}}{y_c A} + x_c$$

where, $I_{xyc}$ is the product of moment of inertia with respect to an orthogonal coordinate system passing through the centroid area and formed by translation of $x - y$ coordinate system. From Eq. (1.2.15), it is clear that the resultant force does not pass through the centroid but always below it. Again, if the submerged area is symmetrical to an axis passing through the centroid and parallel to either $x\ or\ y$ axis, the resultant force must lie along the line $x = x_c$. The point $(x_R, y_R)$ through which the resultant force acts is called as "center of pressure". With the knowledge of engineering mechanics, the centroidal coordinates and moment of inertias for common shapes are given in Fig. 1.2.6.
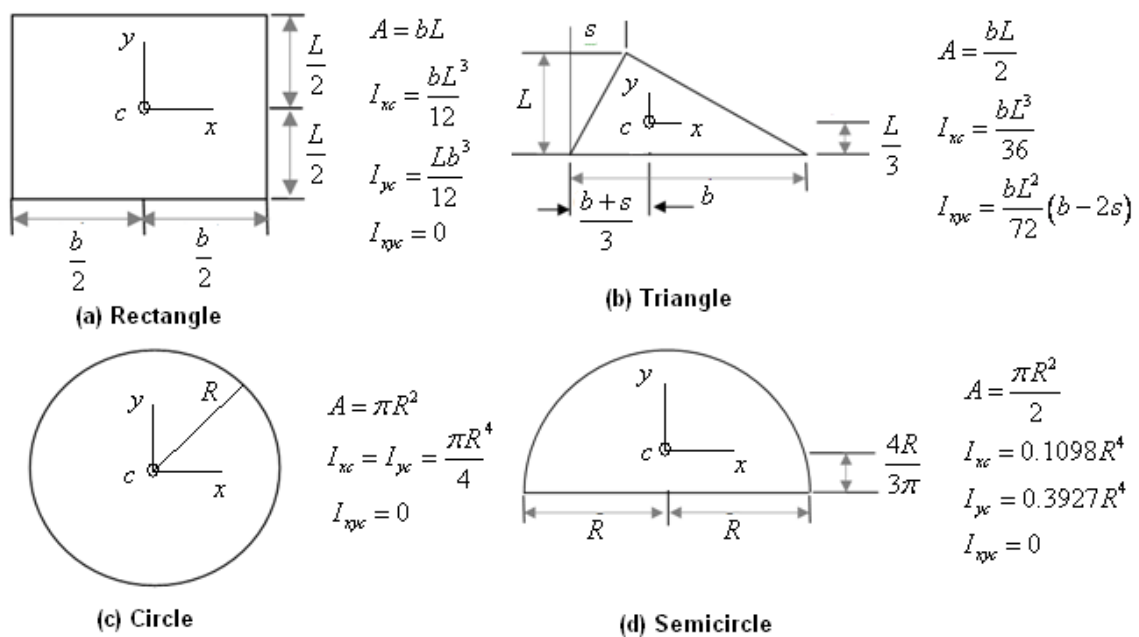


**Fig. 1.2.6: Area and moment of inertia of few common shapes.**