domain

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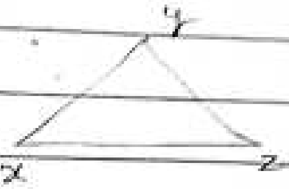
Unit 2

Metric space:- is a distance function where, X is any non empty set such that the function $d : X \times X \longrightarrow \mathbb{R}^+ \cup \{0\}$ is a metric space if satisfies



i) $d(x,y) \geq 0$

ii) $d(x,y) = 0$ iff $x = y$ (null condition)

iii) $d(x,y) = d(y,x)$ (symmetric condition)

iv) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$ Δ inequa-lity

The pair (x,d) is called metric space.

Ex:- 1) If $X = \mathbb{R}$ & d_u &x

$d_u(x,y) = |x-y|$ (usual metric)

then $(x,d) = (\mathbb{R}, d_u)$ is m space

$d(x,y) = |x-y|$

i) $|x-y| \geq 0$ $d(x,y) \geq 0$ (Proved) $|1-2| = |-1| = 1 >0$

ii) $|x-y| = |y-x|$ $\longrightarrow$ $|-(y-x)| = |y-x|$

iii) $|x-y| = 0$ iff $(x=y)$

iv) $|x-y| \leq |x-z| + |z-y|$

$|x-z+z-y|$

$|x-y|$ $|x+y| \leq |x| + |y|$

$|x-y| = |x+z-y-z|$

$= |(x-z) + (z-y)|$

$\leq |x-z| + |z-y|$

$d(x,y) \leq d(x,z) + d(z,y)$

Ex:- 2) $X = \mathbb{C}$, $d_u = |x-y|$ $\Rightarrow$ $x = a+ib$ or can be (a,b)

$y = c+id$ (c,d)

distance formula

$$d(x,y) = |x-y|$$
$$= |a+ib-c-id|$$
$$= |a-c+i(b-d)|$$
$$= \boxed{\sqrt{(a-c)^2+(b-d)^2}}$$

i) $|a-c+i(b-d)| \geq 0$ (obviously)
$\sqrt{(a-c)^2+(b-d)^2} > 0$

ii) $|x-y| = |y-x|$
$|a+ib-c-id|$

iii) $|x-y| = 0$
$x-y=0$
$x=y$

iv) $|z_1+z_2| \leq |z_1|+|z_2|$

Imp Ex'— 3) X is any set, $d(x,y) = \begin{cases} 1 & , x \neq y \\ 0 & , x=y \end{cases}$ discrete metric space (def)

iv) $d(x,y) \leq d(x,z)+d(z,y)$
$1 \leq 1+1$ true when $x \neq y$

case 2. $x=y=3$
$0 \leq 0+0$ true when $x=y$

Ex'- 4) $X = \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$
$d(x,y) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ $\begin{cases} x = (x_1,y_1) \\ y = (x_2,y_2) \end{cases}$

iv) $d(x,y) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
sq.
$$[d(x,y)]^2 = (x_1-x_2)^2+(y_1-y_2)^2$$
$$= (x_1-x_2+x_3-x_3)+(y_1-y_2+y_3-y_3)^2$$
$$= [(x_1-x_3)+(x_3-x_2)]^2+[(y_1-y_3)+(y_3-y_2)]$$

$$= (x_1-x_3)^2 + (x_3-x_2)^2 + 2[(x_1-x_3)(x_3-x_2) + (y_1-y_3)(y_3-y_2)] + (y_1-y_3)^2 + (y_3-y_2)^2$$

If we let $x_1 - x_3 = a$

$x_3 - x_2 = b$

$y_1 - y_3 = c$

$y_3 - y_2 = d$

since $(ab+cd)^2 \leq (a^2+c^2)(b^2+d^2)$

Hence we have $[d(x,y)]^2 \leq (x_1-x_3)^2 + (x_3-x_2)^2 + 2[\sqrt{(x_1-x_3)^2(x_3-x_2)^2} + \sqrt{(y_1-y_3)^2(y_3-y_2)^2}] + (y_1-y_3)^2 + (y_3-y_1)^2$

$$= [d(x,z)]^2 + 2d(x,z)\,d(z,y) + [d(z,y)]^2$$

$$= [d(x,z) + d(z,y)]^2$$

Ex(3) i) $d(x,y) \geq 0 \quad \forall\ x,y \in R$

as either $d(x,y) = 0$ or $d(x,y) = 1$

Proved

ii) If $x = y$, $d(x,y) = 0$ (By defn)

& if $d(x,y) = 0$ then $x = y$

iii) If $x = y$, $d(x,y) = d(y,x) = 0$

If $x \neq y$, $d(x,y) = d(y,x) = 1$

Proved

~~iv)~~

Ex 4) i) To show $d(x,y) \geq 0$

$d(x,y) = \left[(x_1-x_2)^2 + (y_1-y_2)^2 \right]^{1/2}$

here $(x_1-x_2)^2 \geq 0$ & $(y_1-y_2)^2 \geq 0$

Therefore $d(x,y) \geq 0$

ii) To show $d(x,y)=0$ iff $x=y$

$d(x,y)=0$

$\Rightarrow \left[(x_1-x_2)^2 + (y_1-y_2)^2\right]^{1/2} = 0$

$\Rightarrow (x_1-x_2)^2 + (y_1-y_2)^2 = 0$ Sum of 2 +ve nos is zero only when both are eq to zero

$\Rightarrow x_1-x_2=0$ & $y_1-y_2=0$

$\Rightarrow x_1=x_2$ & $y_1=y_2$

$\Rightarrow x=y$

iii) to show $d(x,y) = d(y,x)$

$d(x,y) = \left[(x_1-x_2)^2 + (y_1-y_2)^2\right]^{1/2}$

$= \left[(x_2-x_1)^2 + (y_2-y_1)^2\right]^{1/2}$

$= d(y,x)$

4-8-23 Q) $X = \mathbb{R}^2$

$d(x,y) = |x_1-y_1| + |x_2-y_2|$

where $x=(x_1,x_2)$, $y=(y_1,y_2)$

i, ii, iii) Proved (easily shown)

iv) to show $d(x,y) \leq \cancel{d(x,x) + d(y,z)}$

$d(x,z) + d(z,y)$

let $x = (x_1, x_2)$ $|a+b| \leq |a|+|b|$

$y = (y_1, y_2)$

$z = (z_1, z_2)$

$d(x,y)$ $|x_1-y_1| + |x_2-y_2|$

RHS $|x_1-z_1| + |x_2-z_2| + |z_1-y_1| + |z_2-y_2|$

This matrix is called taxi cab matrice

Q) $X = \mathbb{R}^2$

$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

prove that it's a metric space.

Remark:- The previous 3 example shows that on a non empty set X ~~be~~ we may define more than 1 metric

Ex:- a) $X = \mathbb{R}^n$

same as previous just $X = \mathbb{R}^n$ changed

i) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ where $x = x_1, x_2, \dots, x_n$ & $y = y_1, y_2, \dots, y_n$ → taxi cab metric

" ii) $d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$ usual metric

iii) $d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$, $p \geq 1$

" iv) $d_\infty(x,y) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$ → max metric

$, x_1)$

Pf iii) ~~iv)~~ ~~Minkhov~~ Minkowski's inequality →

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

iv) To prove $d(x,y) \leq d(x,z) + d(z,y)$

$$\left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |\underbrace{(x_i - z_i)}_{x_i} + \underbrace{(z_i - y_i)}_{y_i}|^p\right)^{1/p} \leq$$

$$\left(\sum_{i=1}^{n} |x_i - z_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |z_i - y_i|^p\right)^{1/p}$$

Hence satisfied

☆ bdd seq.
distance will be +ve, 0, & finite (not infinite)

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Ex:- b) Let $X = \mathbb{C}^n$

☆ we define d_1, d_p, d_∞ on X in a similar way as previous example.

Ex:- c) Let l^∞ be the set of all bounded sequence of $\mathbb{R}$ or $\mathbb{C}$ no.

$$\begin{cases} l^\infty = \{ \{x_n\} \subset \mathbb{R} \text{ or } \mathbb{C} : \sup |x_n| < \infty \} \\ d_\infty(x, y) = \sup\limits_{i \geq 1} |x_i - y_i| \end{cases}$$

corresponding seq. minus

$|x_i - z_i + z_i - y_i|$

Ex:- d) Let s be the space of all sequences of $\mathbb{R}$ or $\mathbb{C}$ no.

☆

$$s = \{ \{x\} \subset \mathbb{R} \text{ or } \mathbb{C} \}$$

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

(Frechet metric)

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multiplying $\frac{1}{2^n}$ both sides

$$\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n-y_n|} \leq \frac{1}{2^n}\cdot\frac{|x_n-z_n|}{1+|x_n-z_n|} + \frac{1}{2^n}\frac{|z_n-y_n|}{1+|z_n-y_n|}$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n-y_n|} < \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n-z_n|}{1+|x_n-z_n|} + \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|z_n-y_n|}{1+|z_n-y_n|}$$

Ex:- l^p is the space of all R or C sequences s.t $\sum_{n=1}^{\infty}|x_n|^p <\infty$

$$d(x,y) = \left(\sum_{i=1}^{\infty}|x_n-y_n|^p\right)^{1/p}$$ space of all p summable seq

Ex:- Let $B[a,b]$ be the space of all bdd real value fns defined on closed interval $[a,b]$ i.e

$$B[a,b] = \{ f:[a,b]\to\mathbb{R} : |f(t)| \leq k \ \forall\ t\in[a,b]$$

$$d(f,g) = \sup_{t\in[a,b]}|f(t)-g(t)|$$

Ex:- Let $C[a,b]$ be the space of all conts real value fn defined on $[a,b]$

$$C[a,b] \longrightarrow d_\infty(f,g) = \max_{t\in[a,b]}|f(t)-g(t)| \ \&$$

$$d_1(x,y) = \int_a^b |f(t)-g(t)|\,dt$$

Ex:- Let X be the set of all Riemann integral fns on $[a,b]$ then we define a metric $d(f,g) = \int_a^b|f(t)-g(t)|\,dt$

bdd fn — range set is some bd, upper bd exist

now prove that d is not a metric on X.

open sphere
closed "
Circle define
Pihey "
Disc "
Int. End point

8-8-23 $f(x) = \begin{cases} 0 & \text{if } x = a \\ 1 & o/w \end{cases}$

$f : [a,b] \to R$, $g : [a,b] \to R$

f $g(x) = 1$ $\forall x \in [a,b]$

soln:- discont. at a

$f \in R[a,b]$ d

not a metric

distance b/w sets & diameter of sets:- Let (X, d) be a metric space & let A & B are 2 non empty subs of X. The distance b/w sets A & B

$$\rho(A,B) = \inf\{d(x,y) : x \in A, y \in B\} = \inf\{d(y,x) : y \in B, x \in A\} = \rho(B,A)$$

If A consist of single point, $A = \{x\}$ then

$\rho(A,B) = \rho(\{x\},B) = \inf\{d(x,y) : y \in B\}$

It's called distance of point $x \in X$ from the set B denoted by $\rho(x,B)$.

Ex:- let $A = \{x \in R : x > 0\} = (0, \infty)$

$B = \{x \in R : x < 0\} = (-\infty, 0)$

$\rho(A,B) = 0$

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But A & B have no common point. If $x = 0$ then distance b/w i.e $\rho(x, B) = 0$ but $x \notin B$

Remark:- Distance b/w $\rho(x, B) = 0$ doesn't implies that $x \in B$.

Though $\rho(A, B) = 0 \not\Rightarrow A \cap B \neq \phi$

$d(\phi) = -\infty$

$d(\{x\}) = 0$ → Diameter

Diameter of set:- Let (X, d) be a metric space & A be a non empty subset of X $(A \subseteq X)$. The diameter of A $\delta(A)$ is defined as

$$\delta(A) = \sup \{ \underset{\downarrow}{d(x,y)} : x, y \in A \}$$

usual metric

If discrete metric $d(x,y) = \ldots$

set A is called bdd if $\delta(A) \leq K$ (K → finite no.)

In other words A ~~K~~ is bdd if it's diameter is finite.

Real line with usual metric is an unbdd metric space.

(usual) $d(x, y) = \infty$
where $\delta(A) = \infty$
not bdd

R with discrete metric space is bdd. metric space.

(i) $[a, b]$, (ii) $(a, b]$, (iii) $[a, b)$, (iv) (a, b)

(v) $[a, \infty)$, (vi) $(-\infty, a]$, (vii) (a, ∞), (viii) $(-\infty, a)$ with usual metric space

(i, ii, iii, iv) are bdd $|a - b|$

(v, vi, vii, viii) " not bdd (∞)

Ques) Space of all seq. with $\mathbb{R}$ or $\mathbb{C}$ no. & is bdd one?

$$d(x, y) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

$1 + \frac{1}{2} + \frac{1}{3} \ldots$

$$< \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \qquad \frac{|x_n - y_n|}{1 + |x_n - y_n|} < 1$$

$= 1$

bdd set

☆ Every set with discrete metric is bdd & diameter will be 1.

u) determine distance $d(3,4)$ to the set $[0,1]\times[0,$
in $\mathbb{R}^2$ w.r.t the metric
i) usual
ii) taxi cab $|x - x_1| + |y - y_1|$
iii) max
$1 \leftarrow$ iv) discrete

$|3 - x_1| + |4 - y_1|$
$x_1 = 1, \ y = 1$

09-08-23 Open sets & Interior Points :- set $S \subseteq R$ is said to be open in R if it is nbd of all of it's elements $\forall \alpha \in S$, $\exists$ $\delta > 0$ s.t $(\alpha-\delta, \alpha+\delta) \subset S$.

Open sphere :- Let (X,d) be a metric space given a point $x_0 \in X$ & a R no. $r > 0$, the sets $S_r(x_0) = \{y \in X : d(x_0, y) < r\}$ (y, x_0)

$S_r[x_0] = \{y \in X : d(x_0, y) \leq r\}$

The sets $S_r[x_0]$ & $S_r(x_0)$ are called open sphere (open ball) & closed sphere (closed ball). Respectively with centre at x_0 & radius r.

Remark :- open & closed sphere are always non empty. (x_0 will also belong)

since $S_r[x_0]$, $S_r(x_0)$ will belong that's why ~~Ex~~ every open sphere in R with usual metric is an open interval but converse need not to be true.

Ex :- $(-\infty, \infty)$ is an open interval in R but not an open sphere. (r not defined) x' pt is if ∃ a nbd of x i.e. $(x-\epsilon, x+\epsilon) \subset A$

Ex:- $(\mathbb{R}, d_u)$, $S_r(x_0)$ & $S_r[x_0]$ are intervals
$= (x_0 - r, x_0 + r)$ $= [x_0 - r, x_0 + r]$

Ex:- $(\mathbb{C}, d_u)$, $S_r(z_0)$ & $S_r[z_0]$ are circular disc : $|z - z_0| < r$: $|z - z_0| \le r$

Ex:- Let X be a non empty set (X, d)

$$S_r(x_0) = \begin{cases} \{x_0\} & \text{if } 0 < r \le 1 \\ X & \text{if } r > 1 \end{cases}$$

&

$$S_r[x_0] = \begin{cases} x_0 & \text{if } 0 < r \le 1 \\ X & \text{if } r \ge 1 \end{cases}$$

?

Ex:- Let $X = [0, 1)$ be a metric space with ? usual metric $d(x, y) = |x - y|$

$S_r(0) = \{y \in X : d(0, y) < r\}$ $(r \le 1)$
$= \{y \in X : |0 - y| < r\}$
$= \{y \in X : y < r\}$
$= [0, r)$

when $r > 1$ then $S_r(0) = [0, 1)$

similarly,

$$S_r[0] = \begin{cases} [0, r) & \text{if } r < 1 \\ [0, 1) & \text{if } r \ge 1 \end{cases}$$

Ex:- In $\mathbb{R}^2$, $S_r(0, 1)$ w.r.t the d_1, d_2 & d_∞

$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$
$d_2(x, y) = \left((x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{1/2}$

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$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$x_0=(0,0)$
$r=1$

$S_r(x_0) = \{y \in X : d_1(x,y) < r\}$
$\{y \in \mathbb{R}^2 : d(0,y) < 1\}$

$S_1(0) = \{y \in X : |x_1-y_1|+|x_2-y_2| < 1\}$

$= \{y \in X : |0-y_1|+|0-y_2| < 1\}$

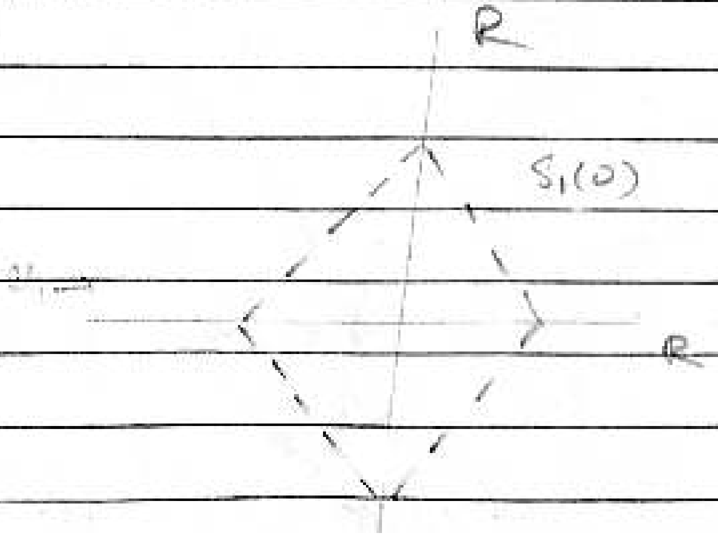
$= \{y \in X : |y_1|+|y_2| < 1\}$
$(y_1,y_2) \in \mathbb{R}^2$

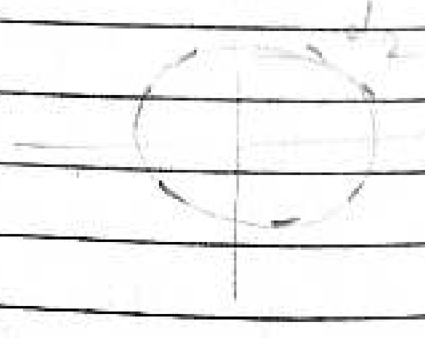
$= \{y \in \mathbb{R}^2 :$

ii) $S_1(0) = \{y \in X : d(0,y) < 1\}$
$= \{y \in X : [(0-y_1)^2+(0-y_2)^2]^{1/2} < 1\}$
$= \{(y_1,y_2) \in \mathbb{R}^2 : (y_1^2+y_2^2)^{1/2} < 1\}$

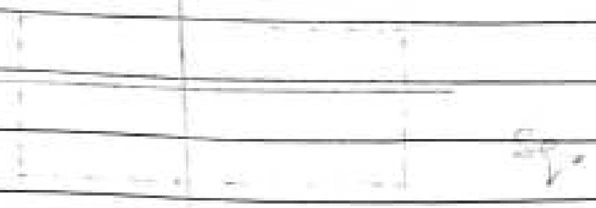
iii)

$\{(y_1,y_2) \in \mathbb{R}^2 : \max\{|y_1|, |y_2|\} < 1\}$





$d_\infty \longrightarrow$



Q) $C[a,b] \longrightarrow$ fn space

$$d(f,g) = \max_{t\in[a,b]} |f(t)-g(t)|$$

$S_r(f_0)$ with centre f_0 & radius +ve is the set of ~~all~~ those continuous functions s.t

$$\max_{t\in[a,b]} |f(t)-g(t)| < r$$

Assig

Q) Check whether following are metric or not

i) $d_1(x,y) = [|x-y|] \longrightarrow$ greatest integer fn

ii) $d_3(x,y) = |\sin(x-y)|$ ×

iii) $d_6(x,y) = |x^2-y^2|$ × $X = \mathbb{R}$

iv) $d_7(x,y) = 2|x-y|$ ✓

v) $d_4(x,y) = e^{|x-y|}$

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Interior Point:- A point $x \in A$ is said to be interior point of A if $\exists$ an open ball centre at "x" & $r > 0$ i.e $S_r(x)$ contained in A

$$x \in S_r(x) \subset A$$

set of all IP of set A is denoted by $\boxed{A^\circ}$ (Interior of 'A')

The set A is said to be open set if all the elements of A is interior point. $\boxed{A = A^\circ}$

set an sabhi element IP hone chahiye

Let $x \in X$, the set A is said to be neighbourhood of "x" if 'A' is an open set.

Remark:- An open sphere $S_r(x)$ with centre x & radius 'r' is a nbd of x

The interior of A° is the nbd of each of it's point

every open set is nbd of each of it's point.

A =

1) The set A of a) (a, b) usual metric of R

R → open sphere (x−r, x+r)

$A^\circ = (a, b)$ nbd of all of it's point

2) If $A = \mathbb{N}$ with usual metric of R

$A^\circ = \phi$ that's why taking interior

Similarly $\mathbb{Z}$ & $\mathbb{Q}$ also has no interior point.

Ex: If $A = (0,1) \cap \mathbb{Q}$ with usual metric $\mathbb{R}$
$A^\circ = \phi$

Ex: If $A = \bigcap [-3, \frac{1}{n})$

$A^\circ = [-3, 0]$ $(-3,1) \cap (-3,\frac{1}{2}) \cap (-3,\frac{1}{3}) \cap (-3,\frac{1}{4}) \cap$

$-3 \quad 0 \quad 1/4 \; 1/3 \; 1/2 \quad 1$

Ex: $A = [-3, 0] \longrightarrow$ not open (usual metric) $[-3,0)$
$A^\circ = (-3, 0)$

$-3 \quad -2 \quad -1 \quad 0$

$-3 \in S_r(x_0) \not\subset A$

Result:- If A is non empty subset of discrete metric space X then $A^\circ = A$ i.e A is open
$[a,b] \longrightarrow$ open (discrete metric)

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

$0 < r$

$S_r(x) = \{y \in X : d(x,y) < r\}$
$= \{x\} \subseteq A \quad 0 < r \leq 1$
$X \quad r \geq 1$

Assignment) Q1) If Pf of above Result
Q2 a) Let $A, B \subseteq (X,d)$ then prove that $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$
b) $(A \cap B)^\circ = (A^\circ \cap B^\circ)$
c) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$ & also prove converse of (c) isn't true i.e $(A^\circ \cup B^\circ) \not\supseteq (A \cup B)^\circ$

Q3) If $A \subseteq (X,d)$ then $(A^\circ)^\circ = A$

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need not to belong

Limit Point :- let $A \subset X$ a point $x \in X$ is said to be lt point (accumulation / cluster point) of A if each open sphere centred on x contains atleast one point of A different from x i.e $\forall\ r > 0$

$$S_r(x) - \{x\} \cap A \neq \phi$$

12-8-23 A' denotes set of all limit points of A & called as derived set.

closed set :- A set A of X is said to be closed if A^c is open.

or

If derived set A' is subset of A then A is closed

$A' \subseteq A$

Ex: If $A = \{1, 1/2, 1/3, \dots\}$

$A' = ?$ in $(R, d) = X \to$ usual metric

$A' = \{0\} \not\subset A$ (not closed)

Ex: If (R, d) $\to$ usual metric

$A = N / Z$ & Q , $A' = \phi$ (closed)

$A' \subseteq A$

Q) $(R, d) \to$ discrete metric

$A = N / Z$

$A' = ?$, $A' = \phi$ (closed)

$$S_r(x) = \begin{cases} \{x\} & 0 < r \le 1 \\ X & r > 1 \end{cases}$$

Ex: Every real no. is lt point of Q

$Q' = R$, $Q^{c\,\prime} = R$

imp theorem :- let (X, d) be a metric space then

a) each open sphere in X is open set.

b) a subset of X is open iff it is the union of open sphere.

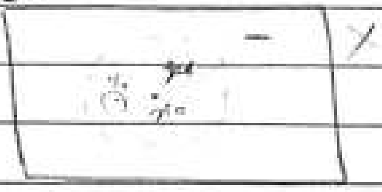
... point of A contained in A i.e $A' \subseteq A$

Date 12/8

Let $S_r(x_0) = \{x \in X : d(x, x_0) < r\}$

a) Pf be a open sphere in X & let $y_0 \in S_r(x_0)$

To prove (i) we have to construct an open sphere centred at y_0 & contained in $S_r(x_0)$. Since $y_0 \in S_r(x_0)$ we have $d(y, x_0) < r$



~~Let~~ $r_1 = r - d(x_0, y_0) > 0$

Consider $S_{r_1}(y_0) = \{y \in X : d(y, y_0) < r_1\}$

Now we have to show that $S_{r_1}(y_0) \in X$

let $y \in S_{r_1}(y_0) \Rightarrow d(y, y_0) < r_1$

$$\therefore d(x_0, y) \leq d(x_0, y_0) + d(y_0, y)$$
$$\leq d(x_0, y_0) + r_1$$
$$\leq d(x_0, y_0) + r - d(x_0, y_0)$$
$$= r$$
$$\Rightarrow y \in S_r(x_0)$$

Since $y \in S_{r_1}(y_0)$ is arbitrary. Hence $S_{r_1}(y_0) \subseteq S_r(x_0)$

Hence every open sphere is an open set.

b) Pf:- Suppose X is an open set then each of it's point is centre of an open sphere contained in A. Hence A is the union of all open spheres contained in A.

conversely :- let us assume that A is the union of a collection F of open sphere. Let $x \in A$ (arbitrary) then $x \in$ some open sphere say, $S_r(x_0) \in F$.

Since each open sphere is a open set, hence x is the centre of an open sphere.

$S_{r_1}(x) \subseteq S_r(x_0) \subset A$

$S_{r_1}(x) \subseteq A$

Hence A is open set.

Result:- Let (X, d) be a metric space then

a) arbitrary union of open set in X is open

b) Finite Intersection of " " " " " "

$\bigcup_{\alpha \in} A_\alpha = A$ Is open (A_α's are open)

$\bigcap_{i=1}^{n} A_i = A$ is open (A_i's are open in X)

countable intersection of open set is need not be o

Ex:- (R, d) is a usual metric in d (not open)

$A_i = (-1/i, 1/i)$

$\bigcap_{i=1}^{\infty} A_i = A = \{0\}$

1 element

infinite elem