19-8-23 **Isolated Point:-** A point $x \in X$ is said to be an " " of A. If $\exists$ an open sphere centred at x containing no point of A other than x itself i.e $\exists$ a $r > 0$: $(S_r(x) - \{x\}) \cap A = \phi$

If a point $x \in X$ isn't a lt of A then it's an isolated point of A.

Ex:- $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with usual metric

then '0' is only lt point of X while all other points are isolated points of X.

Closure of A:- A subset A of X $(\bar{A} \subseteq X)$ is said

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be the closure of $A \subseteq X$ if it is the union of all set of lt points. $\boxed{\bar{A} = A \cup A'}$

Ex:- $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $X = \mathbb{R}$, $d_u = |x-y|$

$\bar{A} = \{0\} \cup A$

Note:- In other words $x \in \bar{A}$ if every open sphere $S_r(x)$ centered at x & $r > 0$ contains a point of A i.e $x \in \bar{A}$ iff $S_r(x) \cap A \neq \phi$, $\forall r > 0$

Result:- Let (X, d) be a metric space & $A \subseteq X$ then $x \in \bar{A}$ iff distance b/w $x, A = 0$ i.e
$$d(x, A) = 0$$

$x \in \bar{A}$
$\Rightarrow x \in A \cup A'$
let $x \in A$
$x \in S_r(x)$,
$x \in A$
$S_r(x) \cap A \neq \phi$

Pf:- Let $x \in \bar{A}$ $\Rightarrow$ If $x \in A$, then $d(x, A) = 0$

$\hookrightarrow d(x, x) = 0$

If $x \notin A$ then $x \in A'$ (x will be lt point of A)
Thus for each $\epsilon > 0$ each open sphere $S_\epsilon(x)$ centered at x & radius ($\epsilon > 0$) intersection with A i.e ($S_\epsilon(x) \cap A$) contains atleast one point i.e 'y'. i.e $d(x, y) < \epsilon$. $y \in S_\epsilon(x)$
since ϵ is chosen arbitrarily +ve. Hence
$$d(x, y) = 0 \Rightarrow d(x, A) = 0 \text{ as } y \in A.$$

Conversely:- suppose $d(x, A) = 0$ if $x \in A$ then $x \in \bar{A}$
When $x \notin A$ then $\because d(x, A) = 0$ & $x \notin A$
$\Rightarrow$ for any $\epsilon > 0$ $\exists$ $y \in A$ s.t $d(x, y) < \epsilon$ i.e $y \in S_\epsilon(x)$
$\Rightarrow$ $S_\epsilon(x) \cap A$ contains y.
$\because$ $x \notin A$ then y can't be eq. to x.
$\therefore$ x is lt point of A.

$\bar{A} = A$

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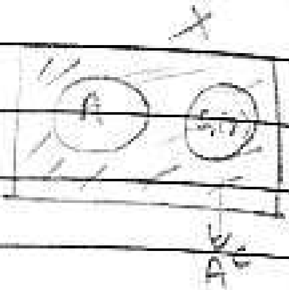
Thus $x \in \bar{A}$.

Thm: A subset 'A' of X is ~~sa~~ closed iff complement A is an open set.

Pf: Let A be closed so we have to prove A^c is open
let $x \in A^c$
Then $x \notin A$ & also x can't be lt point of A $\because$
A is closed, then $\exists_{r>0 \text{ s.t}}$ $S_r(x) \cap A = \phi$

$\Rightarrow$ $S_r(x) \subseteq A^c$



Since x choosen arbitrarily
we get
which is properly contain in

Hence x is complement of A
$\therefore$ A^c is open

Conversely:- Assume that A^c is open.
Let $x \in X$ be a lt point of A if $x \in A$ then
A contains all it's lt point.
Hence A is closed.
If $x \notin A$ $\Rightarrow$ $x \in A^c$
$\because$ A^c is open then $\exists$ ~~an open~~ $S_r(x)$ s.t
$S_r(x) \subseteq A^c$
Consequently $S_r(x) \cap A = \phi$ for some $r > 0$
Hence x can't be a lt point which contradicts our assumption.
$\therefore$ $x \in A$
$\Rightarrow$ A is closed? open

(using previous theorem)

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Theorem:- In a metric space (X, d) every closed sphere is closed set.

Pf:- $S_r[x] = \{ y \in X : d(x,y) \leq r \}$

Let $S_r[x]$ is closed set then it's sufficient to prove that $(S_r[x])^c$ is open. $A^c = A$

Taking any point $y \in (S_r[x])^c$

then $y \notin S_r[x]$

$\therefore$ $d(x,y) > r$ (because $y \notin S_r[x]$ can't be $\leq r$) → will show int point

Now set

$r_1 = d(x,y) - r > 0$

Now let $z \in S_{r_1}(y)$ then $d(z,y) < r_1$

By Δ inequality $d(x,y) \leq d(x,z) + d(z,y)$

$$\Rightarrow d(x,z) \geq d(x,y) - d(z,y)$$
$$> d(x,y) - r_1$$
$$= r$$

$\therefore$ $z \notin S_r[x]$

Hence $z \in (S_r[x])^c$

Thus $S_{r_1}(y) \subseteq (S_r[x])^c$

$\Rightarrow$ y is interior point of $(S_r[x])^c$ & y was arbitrary.

Hence $(S_r[x])^c$ is open $\Rightarrow$ $S_r[x]$ is closed

De - Morgan's Law :- $\bigcap_{\alpha \in A} (A_\alpha^c) = \left(\bigcup_{\alpha \in A} A_\alpha\right)^c$

& $\bigcup_{i=1}^{n} A_i^c = \left(\bigcap_{i=1}^{n} A_i\right)^c$

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arbitrary union of closed sets need not be closed.

Ex:- $\{[1/n, 2] : n \in \mathbb{N}\}$ in usual metric $(\mathbb{R}, d)$

$\bigcup_{n\in\lambda}\{[1/n, 2] : n \in \mathbb{N}\} = (0, 2]$ which is

not closed in (R, d)

23 Exterior Point :- A point $x \in X$ is said to be an ext point of $A \subseteq X$ if it (x) is an int point of A^c.

Boundary Point:- A point $x \in X$ is said to be boundary point of $A \subseteq X$ if x is neither an int point nor a ext point of A

Ex: $A = [a, b]$ $\quad X = (\mathbb{R}, d)$

$b(A) = \{a, b\}$ $\quad Ex(A) = (-\infty, a) \cup (b, \infty)$

$A^\circ = (a, b)$

$A^c = (-\infty, a) \cup (b, \infty)$ $\quad \rightarrow X - (A^\circ \cup Ex(A))$

$(A^c)^\circ = (-\infty, a) \cup (b, \infty)$

$A = \mathbb{N}$, $X = (\mathbb{R}, d)$

$A^\circ = \phi$ $\quad (-\infty, 0) \cup \mathbb{N}$

$A^c = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots = \mathbb{R} - \mathbb{N}$

$(A^c)^\circ = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots = \mathbb{R} - \mathbb{N}$

$b(A) = \mathbb{R} - (\phi \cup (\mathbb{R} - \mathbb{N})) = \mathbb{N}$ $\quad (\mathbb{R} - (\mathbb{R} - \mathbb{N}))$

$A = \phi$, $X = (\mathbb{R}, d)$

$A^\circ = \phi$ $\quad b(A) = \mathbb{R} - \phi = \cancel{\phi}\ \mathbb{R}$

$A^c = \mathbb{R} - \phi = \mathbb{R}$ $\quad Ext(A) = \phi$

$(A^c)^\circ = \phi$

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If $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ $\quad X = (R, d)$

$A^\circ = \phi$

$A' = 0$

$A^c = R - A = (-\infty, 0] \cup \left(\bigcup_{n=1}^{\infty}\left(\frac{1}{n}, \frac{1}{n+1}\right)\right) \cup (1, \infty)$

$Ext(A) = (A^c)^\circ = (-\infty, 0) \cup \left(\bigcup_{n=1}^{\infty}\left(1/n, 1/n+1\right)\right) \cup (1, \infty)$

$b(A) = R - (A^\circ \cup Ex(A)) = \{0, 1, 1/2, 1/3, \dots\}$

Exc:- $(X, d) \to$ discrete

$A \subseteq X$ $\qquad d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

$b(A) = ?$

$A^\circ = \cancel{\phi}\ A$ $\qquad \{x\}\ r \leq 1$

$A^c = \cancel{X} - \phi = A$ $\qquad X \quad r \geq 1$

$(A^c)^\circ = X - A$

$b(A) = \phi = X - (A \cup A^\circ) = X - X$

Prove that every singleton set & every finite set in MS are closed.

Sol:- $A = \{x\}, \subseteq (X, d)$ $\qquad A = \{0\}$

$A' \subset A$ $\qquad A' = \phi$

Let 'l' be a lt point of A

If $l \in A$ i.e $l = x$

If we take an open ball of radius $r > 0$ centre at 'x' i.e $S_r(l)$, then we have

$$(S_r(l) - \{l\}) \cap A = \phi$$

(finite/infinite) $\cap$ finite = finite

$\Rightarrow l$ isn't lt point of A $\qquad$ But removed l which belongs to A

$A' = \phi$ $\qquad$ That's why $= \phi$

$\phi = A' \subset A$

Every finite set in MS (X,d) is closed.

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Ques) Prove that any closed subset A of a metric space (X,d) is a countable intersection of open set.

For each $n \in \mathbb{N}$, let $O_n = \bigcup_{x \in A} S_{1/n}(x)$

Then O_n is open & $A \subset O_n \quad \forall\, n \in \mathbb{N}$

$\therefore\ A \subseteq \bigcap_{n \in \mathbb{N}} O_n$

We claim that $A = \bigcap_{n \in \mathbb{N}} O_n$ $\rightarrow$ or $(O_n \supseteq A)$

Assume contrary that $\exists\ y \in (\bigcap_{n \in \mathbb{N}} O_n - A)$

Since A is closed, A^c is open & $y \in A^c$

Hence $\exists\ \epsilon > 0$ s.t $S_\epsilon(y) \subseteq A^c$

Since $\epsilon > 0$, $\exists\ n \in \mathbb{N}$ s.t $1/n < \epsilon$ Archimedean Prop.

Since $y \in O_n$, $\exists\ x \in A$ s.t

$d(x,y) < 1/n < \epsilon$, so, $S_\epsilon(y) \cap A \neq \phi$

$\Rightarrow A \cap A^c \neq \phi$

which is contradiction

Pf:- Let A be a non empty subset of MS X. Prove or answer the following statement

a) $b(A)$ is closed set

b) $b(A) = b(X \setminus A) = b(A^c)$

c) If $x \in b(A)$ does x have to be a lt point?

d) $x \in b(A)$ iff for every $\epsilon > 0$ $S_\epsilon(x)$ contains points of A & $A^c (X \setminus A)$

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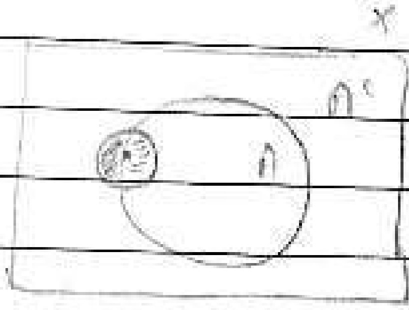
a) since closure of any set is a closed set & $b(A) = \bar{A} \cap \bar{A^c}$
since finite intersection of closed set is closed
hence boundary of is closed.

b) To prove :- $b(A) = b(X \setminus A) = b(A^c)$
Pf :- To show $b(A) \subset b(X \setminus A)$
let $x \in b(A)$ then x will not be int. & ext. point of A
$\Rightarrow$ x will not be int. & ext. point of A^c
$\Rightarrow x \in b(A^c)$
$b(A) \subset b(X \setminus A)$
Now to show $b(X \setminus A) \subset b(A)$

$\Rightarrow$ x will not be int. & ext. point of

c) $A = \{0\} \subseteq (\mathbb{R}, d)$
$b(A) = \{0\}$
$A^\circ =$
No, 0 is not lt point of A

d) let $x \in b(A)$
since $b(A) = \bar{A} \cap \bar{A^c}$, we have
$x \in \bar{A}$ & $x \in \overline{(X \setminus A)}$, then
x is lt point of A as well
as of $X \setminus A$ then for any $\epsilon > 0$, we have $S_\epsilon(x) \cap A \neq \phi$
& $S_\epsilon(x) \cap A^c \neq \phi$



lt point of

Conversely :- let $x \in X$ s.t. for any $\epsilon > 0$ $S_\epsilon(x) \cap A \neq \phi$
& $S_\epsilon(x) \cap A^c \neq \phi$
It is enough to show that $x \in \bar{A}$ if $x \in A$ then $x \in \bar{A}$

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Assume $x \notin A$. let $\epsilon > 0$, WKT $S_\epsilon(x) \cap A \neq \phi$ since $x \notin A$, then $S_\epsilon(x) \cap A$ contains a point of A other than x. Hence x is a lt point of A i.e $x \in \bar{A}$.

$\bar{A} = A \cup A'$

Subspace :- Let (X, d) be a metric space & Y be a subset of X. The relative " d_Y on Y is the restriction of metric fn d on $Y \times Y$ i.e $d_Y(x,y) = d(x,y)$ $\forall$ $x, y \in Y$.
It's easy to see that d_Y is metric on Y. The space (Y, d_Y) is called metric subspace of metric space (X, d)

Ex:- Let $\mathbb{R}$ be usual metric space then
$d(x,y) = |x-y|, x,y \in \mathbb{R}$ $Y = [0,1]$ or $(0,1)$ or $[0,1)$ or $(0,1]$
& $d_Y(x,y) = |x-y|$, $x, y \in Y$
$(Y, d_Y) \rightarrow$ metric subspace

Ex:- Let $(\mathbb{R}, d_u)$ & $Y = \mathbb{Q}$ then define
$d_Q : Q \times Q \rightarrow R^+ \cup \{0\}$
s.t $d_Q(x,y) = |x,y| = d(x,y)$. $\forall x, y \in Q$
$(Q, d_Q) \rightarrow$ metric subspace of MS (R, d)

$d_\infty = \max |f(t) - g(t)|$ $\forall f, g \in P[a,b]$

Ex:- Let $Y = P[a,b]$, set of all polyn define on $[a,b]$
now define $d_p : P[a,b] \times P[a,b] \rightarrow R^+ \cup \{0\}$
s.t $d_p(f,g) = \max_{t \in [a,b]} |f(t) - g(t)| = d_\infty(f,g)$

$\rightarrow$ $X = C[a,b]$, $d_\infty = \max_{t \in [a,b]} |f(t) - g(t)|$ $\forall f, g \in C[a,b]$

$(P[a,b], d_p)$ metric subspace of metric space $(C[a,b], d_\infty)$

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Thm:- Let (Y, d_Y) be a subspace of a MS
If $a \in Y$, $r > 0$ then $S_r'(a) = Y \cap S_r(a)$
where $S_r'(a)$ are open spheres in (X, d) & (Y, d_Y) respectively

Pf:- we have $S_r(a) \cap Y = \{x \in X : d(x,a) < r\} \cap Y$
$= \{x \in Y : d(x,a) < r\}$ $\because Y \subseteq X$
$= S_r'(a)$

Theorem :- Let (Y, d_Y) be subspace of a MS (X, d)

9-9-23 & A, a subset of Y, then

a) A is open in Y iff $\exists$ an open set G in X s.t $A = G \cap Y$

b) A is closed iff $\exists$ a closed set F in X s.t $A = F \cap Y$ in Y

Pf:- Let $S_r(x)$ & $S_r'(x)$ are same as pre result.
Suppose that $A = G \cap Y$ where G is open in X.
Let $x \in A$ be arbitrary. Then we have to do that x is an int. point of A.
Since $A = G \cap Y$ & $x \in A$, we have $x \in G$ & $x \in Y$
$\because$ G is open in X, there exist $r > 0$ s.t $S_r(x) \subset G$. Hence by previous result we have
$S_r'(x) = S_r(x) \cap Y \subseteq G \cap Y = A$
$\Rightarrow$ x is an int. point of A w.r.t d_Y
$\Rightarrow$ A is open in Y.

Conversely
Assume that A is open in Y & let $x \in A$ be

arbitrary, then $\exists$ an open sphere $S'_{r}(x)$ of $S_{r_x}(x)$
$S'_{r}(x) \subset A$, Now we have $A = \bigcup_{x \in A} S'_{r_x}(x) = \bigcup_{x \in A} (S_{r_x}(x) \cap Y)$

$= \bigcup_{x \in A} S_{r_x}(x) \cap Y$

$= G \cap Y$

where $G = \bigcup S_{r_x}(x)$
but G be an arbitrary union of open spheres in X is open set in X.
Hence $A = G \cap Y$ where G is an open set in Y

B) A is closed in $Y \iff (Y \setminus A)$ is open in Y
iff $(Y \setminus A) = G \cap Y$ for some open set G of X

iff $A = Y \setminus (G \cap Y)$
$\iff A = (X \cap Y) \setminus (G \cap Y)$
" $A = (X \setminus G) \cap Y$, closed
$A = F \cap Y$ where $F = X \setminus G$

since G is open in X
Hence F is closed in X

~~previous semester proof~~

Corollary :- $(Y, d_Y) \to$ subspace of (X, d) & A be a subset of X.

a) If A is open in Y & Y is open in X then A is open in X.

b) If A is closed in Y & Y is closed in X, then A is closed in X.

imp Theorem:- Let (Y, d_Y) be a subspace of MS (X, d)

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If $A \subset Y$ then i) $x \in Y$ is a lt point of A in Y iff x is a lt point of A in X.

ii) the closure of A in Y denoted by $cl_Y(A)$ is $cl_X(A) \cap Y$, where $cl_X(A)$ is the closure of A in X. In other words $cl_Y(A) = cl_X(A) \cap Y$

Pf:- i) Let $x \in Y$ be the lt point of A in Y
then for every open sphere $S_r(x)$ in Y we have $(S'_r(x) - \{x\}) \cap A \neq \phi$
Now for any given $r > 0$, we have ~~$S_r(x$~~
$(S_r(x) - \{x\}) \cap A = (S'_r(x)^{\cap Y} - \{x\}) \cap A \quad (\because A \subseteq Y)$

$= (S'_r(x) - \{x\}) \cap A \neq \phi$

$\Rightarrow x$ is lt point of A in X

converse can be established by retracting the above steps ?

ii) $cl_Y(A) = cl_X(A) \cap Y$

$cl_X(A)$ is closed in X
Hence by previous theorem $cl_X(A) \cap Y$ is closed in Y.
Since $cl_X(A) \cap Y$ contains A & since $cl_Y(A)$ is the intersection of all closed subsets of Y containing A.
We must have ~~closure of A~~ $cl_Y(A) \subseteq cl_X(A) \cap Y$
On the other hand, $cl_Y(A)$ is closed in Y then
$cl_Y(A) = F \cap Y$ where F is closed set in X by previous result. Since $A \subseteq cl_Y(A)$, F is closed in X containing A.

since $cl_X(A)$ is the intersection of all closed sets containing A we have $cl_X(A) \subseteq F$.

Hence $cl_X(A) \cap Y \subseteq F \cap Y$
$= cl_Y(A)$

$\Rightarrow cl_X(A) \cap Y = cl_Y(A)$

—x—x—x—x—x—x—x—

$|x_n - l| < \epsilon$

UNIT-3

12-9-23 Convergent sequence :- A seq. x_n in (X,d) is said to be convergent to some limit $L \in$ if $\forall\ \epsilon > 0\ \exists\ m \in \mathbb{N}$ s.t $d(x_n, L) < \epsilon$ for $\forall\ n \geq m$

EX:- $x_n = \{1/n\}$ be a seq. in $\mathbb{R}$ with $(\mathbb{R}, d)$
$\{x_n\} \to 0$ (as $n \to \infty$)

EX:- If we consider the metric space $X = (0,1)$ & $d \to$ usual metric
$\{x_n\} = \{\frac{1}{n}\} \longrightarrow 0 \notin X$

So x_n isn't convergent in X.

EX:- Let $\{f_n\}$ be a seq. in $C[0,1]$

$f_n(t) = e^{-nt}$ then $f_n \to 0$ w.r.t metric d_1 on $C[0,1]$ as,

$$d_1(f_n, 0) = \int_0^1 |e^{-nt} - 0|\, dt$$

$$= \int_0^1 e^{-nt}\, dt = \left[\frac{e^{-nt}}{-n}\right]_0^1$$

$\forall \epsilon > 0 \; \exists m \in \mathbb{N}$ s.t $|x_n - l| < \epsilon$ for $\forall n \geq m$

$f : \mathbb{N} \to \mathbb{R}$
$f : \mathbb{N} \to X$ (seq. of X)

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$$\frac{e^{-nt}}{-n} + \frac{1}{n} = \frac{1}{n}\left(1 - e^{-nt}\right) \longrightarrow 0 \quad \text{as } n \to \infty$$

$f_n \to 0$ (function) w.r.t d_1

On the other hand same seq. isn't convergent w.r.t $d_\infty(f, g) = \max\limits_{t \in [a,b]} |f(t) - g(t)|$

Ex:- $d_\infty(f_n, 0) = \max\limits_{t \in [0,1]} |e^{-nt} - 0| = e^{-nt} \longrightarrow 1 \neq 0$ as $n \to \infty$

Hence $\{f_n\} \not\to 0$ w.r.t d_∞

Definition :-
A seq. $\{x_n\}$ in MS (X, d) convergence to a point L of X iff for each $\epsilon > 0$ $\exists$ $m \in \mathbb{N}$ s.t $x_n \in S_\epsilon(L)$, $\forall n \geq m$.

1) Remark :- An eventually constant seq. of a constant seq. is always convergent.

Imp 2) Remark :- In a discrete MS a seq. can converge to a point only if it is an eventually constant seq.

Sequence → A seq is a mapping from set of $\mathbb{N}$ to a space X
$f : \mathbb{N} \to X$. then we say this seq. is a seq. of X.

A seq $\{x_n\}$ is said to be eventually constant seq if $\exists$ a $\mathbb{N}$ s.t $x_n = a$ $\forall n \geq m$

Ex) $x_n = \begin{cases} 1 & \text{for } n = 1 \text{ to } 4 \\ 2 & \text{o/w} \end{cases}$ $m = 5$

Ex) $x_n = \begin{cases} 1 & n = 1 \text{ to } 4 \\ 3 & n = 4 \text{ to } 5000 \\ \rightarrow & \text{o/w} \end{cases}$ $n = 5001$

Theorem:-

~~In a~~ MS (X, d)

Remark 2) Pf:-

$$d(x, y) = \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases}$$

$\{x_n\} \longrightarrow l$

$\forall \epsilon > 0 \; \exists \; m \in \mathbb{N}$ s.t $d\cancel{(x, y)}$

$d(x_n, l) < \epsilon, \; \forall n \geq m$

$d(x_n, l) = 0 \quad \forall n \geq m$

$x_n = l \quad \forall n \geq m$

$x_n = \begin{cases} f_n & , n = 1 \text{ to } m-1 \\ l & , \text{o/w} \end{cases}$

Range set:- $\{ f(1), f(2) \ldots, f(m-1), l \}$

1) Result:- The lt of a seq. in MS (X, d) is unique.

2) Result:- A seq. is convergent in MS (X, d) then seq. will be bdd.

Bdd seq:- A seq. in MS is said to be bdd possible values of seq.

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If $\underbrace{\text{Range set}}$ of seq. is bdd.

13-9-23 Let $\{x_n\}$ be a seq in (X,d)

if let $\{x_n\} \to l_1$ as $n \to \infty$

$\{x_m\} \to l_2$ as $n \to \infty$

By def.

for any $\epsilon > 0$ $\exists$ m_1 s.t $d(x_n, l_1) < \frac{\epsilon}{2}$ $\forall n \geq m_1$

$\exists$ m_2 s.t $d(x_n, l_2) < \frac{\epsilon}{2}$ $\forall n \geq m_2$

$\max\{m_1, m_2\} = m$

then $d(l_1, l_2) \leq d(l_1, x_n) + d(x_n, l_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

x_n, l_1

$< \epsilon$ $\forall n$

Result 2) Pf:-

let $\{x_n\} \to x$ in (X,d)

$\forall \epsilon > 0$, $\exists$ $m \in N$ s.t $d(x_n, x) < \epsilon$, $\forall n \geq m$

In particular $\epsilon = \frac{1}{2}$ then we have $N \in N$ s.t

$d(x_m, x) < 1/2$ $\forall n \geq \cancel{m}\ N$

let $r = \max\{d(x_n, x) : n = a \text{ to } N-1, 1/2\}$ $\forall n$

$m \in N$ $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$

$\leq r + r$

$= 2r$

Theorem:- Let (X, d) be MS. If $\{x_n\}$ & $\{y_n\}$ be seq. in X s.t $x_n \to x$ & $y_n \to y$ then prove that the metric function is continuous.

$x_n \to x$

$f(x_n) \to f(a)$

$x_n \longrightarrow x$

$\forall \epsilon > 0$, $\exists$ $m_1 \in \mathbb{N}$ s.t $d(x_n, x) < \frac{\epsilon}{2}$ $\forall n \geq m_1$

$y_n \longrightarrow y$

$\forall \epsilon > 0$ $\exists$ $m_2 \in \mathbb{N}$ s.t $d(y_n, y) < \frac{\epsilon}{2}$ $\forall n \geq m_2$

$$|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)|$$

$$\leq |d(y_n, y)| + |d(x_n, x)|$$

$$= \epsilon \quad \forall n \geq m$$

Cauchy seq:- A seq. $\{x\}$ in MS (X, d) is said to be a cauchy seq. if for each $\epsilon > 0$ $\exists$ a +ve N s.t $d(x_m, x_n) < \epsilon$, $\forall n, m \geq N$.

Ex:- Let $X = C[0,1]$, $d_\infty(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$

Ex:- The seq. $\{f_n\}$ in X is given by $f_n(t) = \frac{nt}{n+t}$ $\forall t \in [0,1]$ is a cauchy seq. in X.

Indeed for $m \geq n$, the f^n $t \to \frac{mt}{m+t} - \frac{nt}{n+t}$

$$= \frac{(m-n)t}{(m+t)(n+t)}$$

being cont. on $[0,1]$ assumes it's max. value at some point $t_0 \in [0,1]$.

So $d_\infty(f_m, f_n) \leq \max_{t \in [0,1]} |f_m(t) - f_n(t)|$

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$$= \frac{(m-n)t_0^2}{(m+t_0)(n+t_0)} \leq \frac{t_0^2}{(n+t_0)} \leq \frac{1}{n} \to 0$$ for large value of m & n

Moreover the seq. $\{f_n\}$ converges to some lt
Indeed let $f(t) = t$ then
~~$|f_n - f| < \epsilon$~~ $d_\infty(f_n(t), f(t)) = \left|\frac{nt}{n+t} - t\right|$

$$= \left|\frac{nt - tn - t^2}{n+t}\right|$$

$$= \left|\frac{t^2}{n+t}\right| \leq \frac{1}{n} \to 0$$

Hence $\{f_n\} \to f \quad \forall \; t \in [0,1]$

Theorem:- Every convergent seq. in MS is cauchy seq.

Pf: Let (X,d) be a MS & $\{x_n\}$ be a seq. in X such as $\{x_n\} \to x$ then $\forall \; \epsilon > 0 \; \exists$ $m \in \mathbb{N}$ s.t $d(x_n, x) < \epsilon$, $\forall \; n \geq N$.

$x \in (X,d)$ ↓ conv

In particular we can say that $d(x_n, x) < \frac{\epsilon}{2}$ $\forall \; n \geq m$ (N).

By Δ inequality $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall \; m,n \geq N$$

$$= \epsilon \quad \forall \; m,n \geq N$$

Remark:- converse of above need not to be true

Ex:- consider seq. $\{x_n\}$ is usual metric space

point c space
convergence different from convergent

$(\mathbb{Q}, d)$, $d(x,y) = |x-y|$, $\forall x,y \in \mathbb{Q}$

where $x_1 = 1.4$

$x_2 = 1.41$

$x_3 = 1.414$

$x_4 = 1.4142$

$x_5 = 1.41421$

- - - - - - - .

- - - - - - - .

Then seq. $\{x_n\} \longrightarrow \sqrt{2} \in \mathbb{Q}^c$

Hence " isn't convergent.

However seq. is cauchy.

\# seq. $\{x_n\} \longrightarrow \sqrt{2}$, hence seq. is cauchy seq. However it doesn't converge to a point of $\mathbb{Q}$ hence this seq. isn't convergent.

Self :- # $\frac{1}{n}$ is cauchy.

$1/n$ in $((0,1), d)$ not convergent $0 \notin (0,1)$

\# $1-\frac{1}{n}$ ... cauchy

$1-1/n \rightarrow$ conv. in $((0,1), d)$ not conv. in $(0,1)$

19-9-23 **Complete Space**

A MS (X, d) is said to be complete if every cauchy seq. in X converges to a point in X.

Remark :- The usual metric space are $\mathbb{R}$ & $\mathbb{C}$ are complete MS.

Ex 1) ℓ^p $(1 < p < \infty)$

$$d_p(x,y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{1/p}$$

is a complete MS

$x - x_n \in l_p$

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n'ndia

Pf:- Let $\{x^{(m)}\}_{m\in N}$ be a cauchy seq. in l^p where $x^{(m)} = \{x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \dots\} = \{x_k^{(m)}\}_{k\in N}$ seq of seq

s.t $\sum_{k=1}^{\infty} |x_k^{(m)}|^p < \infty \quad \forall \; m = 1, 2, \dots$

then for each $\epsilon > 0$ $\exists$ +ve integer N

s.t $d_p(x^{(m)}, x^{(n)}) = \left(\sum_{k=1}^{\infty} |x_k^{(m)} - x_k^{(n)}|^p\right)^{1/p} < \epsilon$

$\forall \; m, n \geq N$

$\longrightarrow (2.1)$

& thus $|x_k^{(m)} - x_k^{(n)}|^p < \epsilon^p$

$\Rightarrow |x_k^{(m)} - x_k^{(n)}| < \epsilon \quad \forall \; m, n > N$ & $\forall \; k = 1, 2, \dots$

$\Rightarrow$ for each fixed k $\{x_k^{(m)}\}$ is a cauchy seq. in $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) seq. of no.

since $\mathbb{K}$ is complete $\therefore$ it converges in $\mathbb{K}$.

Let $x_k^{(m)} \rightarrow x_k$ as $m \rightarrow \infty$, $\forall \; k$.

Using these limits, we define $x = (x_1, x_2, \dots)$ & show that $x \in l^p$ & $x^{(m)} \rightarrow x$

From 2.1 we have

finite sum (small) $\leftarrow$ $\sum_{k=1}^{l} |x_k^{(m)} - x_k^{(n)}|^p < \epsilon^p \quad \forall \; m, n > N$ for any $l = 1, 2, \dots$

Letting $n \rightarrow \infty$, we obtain,

$$\sum_{k=1}^{l} |x_k^{(m)} - x_k|^p < \epsilon^p \quad \forall \; m > N \text{ \& for any } l = 1, 2, \dots$$

$\left\{\sum_{k=1}^{l} |x_k^{(m)} - x_k|^p\right\}_{l \geq 1}$ is a monotonic

Increasing seq. which is bdd above &

$\therefore$ has a finite lt

$$\sum_{k=1}^{\infty} |x_k^{(m)} - x_k|^p \leq \epsilon^p \quad \forall \quad m > N \qquad (2.2)$$

By Minkowski's Inequality

$$\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k^{(m)} - x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |x_k^{(m)}|^p\right)$$

$$< \epsilon^p + E$$

Hence $x \in l^p$

Moreover from (2.2) we obtain

$d_p(x^{(m)}, x) < \epsilon \;, \forall \; m > N$

$\Rightarrow \quad x^{(m)} \longrightarrow x \quad \& \; x \in l^p$

$\therefore$ Space in l^p in given metric is a complete metric space.

(A) 1:-

Ex:- The space R^n or C^n with usual metric is complete metric space.

$$d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

(A) 2:-

Ex:- The space l_∞ with max. metric is complete metric space.

(A) 3:-

Ex:- The set of Integers Z with usual metric is a complete metric space.

20-9-23 C[a, b] with max-metric is complete metric space.

$$d_\infty(x,y) = \max_{t \in [a,b]} |x(t) - g(t)|$$

f, g ; f

Pf: C[a, b] ~~let~~ Let $(f_m)_{m \in N}$ be a Cauchy seq in C[a, b]. Then for each $\epsilon > 0$ $\exists$ a Natural no. N s.t

seq. of cont. fn

$$d_\infty(f_m, f_n) < \epsilon \quad \forall \; m, n > N$$

or

$$\max_{t \in [a,b]} |f_m(t) - f_n(t)| < \epsilon \quad " \qquad \text{—— (A)}$$

Then for any fixed $t_0 \in [a, b]$ we have

$$|f_m(t_0) - f_n(t_0)| < \epsilon \quad \forall \; m, n > N$$

$\{f_m(t_0)\}_{m \in N}$ is a Cauchy seq. in R.

since R is complete.

This seq. converges.

Let $\{f_m(t_0)\} \longrightarrow f(t_0)$ as $m \longrightarrow \infty$

In this way we can associate to each $t \in [a, b]$ a unique real no. $f(t)$

Then we have defined a fn f with domain $[a,b]$.

Now we show that $f \in C[a,b]$ & $f_m \to f$

from eqn (A) we obtain

$$|f_m(t) - f_n(t)| < \epsilon \quad \forall \; m,n > N \; \& \; \forall \; t \in [a,b]$$

letting $n \to \infty$

$$|f_m(t) - f(t)| < \epsilon \quad \forall \; m > N \; \& \; \forall \; t \in [a,b] \quad \text{—— (B)}$$

To show that f is contn, we consider any $t_0 \in [a,b]$ & any $\eta > 0$

then we have,

$$|f_m(t) - f(t)| < \frac{\eta}{3} \quad \forall \; n > N_1(\eta) \; \& \; \forall \; t \in [a,b]$$

Let $n \geq N_1(\eta)$, then

$$|f_n(t) - f(t)| < \frac{\eta}{3} \quad \forall \; t \in [a,b] \quad \text{—— (D)}$$

By continuity of f_n, $f_n t$ we obtain $\delta > 0$ s.t

$$|f_n(t) - f_n(t_0)| < \frac{\eta}{3} \quad \text{whenever } |t - t_0| < \delta \quad \text{—— (C)}$$

Now from C & D

$$|f(t) - f(t_0)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(t_0)|$$

$$+ |f_n(t_0) - f(t_0)|$$

$$< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3}$$

$$= \eta \quad \text{whenever } |t - t_0| < \delta$$

Hence $f \in C[a,b]$

Moreover from (B) we have

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$$d_\infty(f_m, f) = \max_{t \in [a,b]} |f_m(t) - f(t)| < \epsilon \quad \forall m > N$$

$\Rightarrow f_m \longrightarrow f$ as $m \longrightarrow \infty$

Hence $C[a,b]$ is complete MS.

Remark:- The space $C[0,1]$ is not complete w.r.t metric $d_1(f,g) = \int_0^1 |f(t) - g(t)| \, dt$

Assignment

Q) Prove that space of all $\mathbb{N}$ with metric $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \; \forall x, y \in \mathbb{N}$ isn't complete.

Pf:- Let $\{n\}_{n \in \mathbb{N}}$ in $\mathbb{N}$. Let $\epsilon > 0$ & N be the least Integer $(N > \frac{1}{\epsilon})$. If $m, n > N$ then

$$d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \max\left\{\frac{1}{m}, \frac{1}{n}\right\} < \frac{1}{N} < \epsilon$$

greater integer (m, n) $\leq N$

$\Rightarrow \{n\}$ is cauchy

Suppose on contrary that $\{n\} \longrightarrow p \in \mathbb{N}$

Let $N_1 \longrightarrow$ any Integer greater than $2p$.

Then $n \geq N_1$

$$\Rightarrow d(p,n) = \left|\frac{1}{p} - \frac{1}{n}\right| = \left(\frac{1}{p} - \frac{1}{n}\right) \geq \left(\frac{1}{p} - \frac{1}{N}\right) > \left(\frac{1}{p} - \frac{1}{2p}\right)$$

This shows that seq $\{n\} \not\longrightarrow p$

$\therefore$ Contradiction.

our assumption was wrong.
Hence (N, d) isn't . . . .

Ex:- set of all polyn $P[a,b]$ with uniform metric
$d_\infty(f,g) = \max_{t\in[a,b]} |f(t)-g(t)|$ isn't complete
$P[a,b] \subseteq C[a,b]$

Dense set:- A subset $A \subset X$ is said to be dense in X if $\bar{A} = X$

or

For any $x \in X$ $\exists$ a $y \in A$, $d(x,y) < \epsilon$, where ϵ is any +ve $\mathbb{R}$ no.

Ex:- set of Q is dense in R with usual metric imp
$\bar{Q} = R$

~~Separable space:- If A subset A of X is said to be~~

Separable space:- A set X is said to be separable if $\exists$ a countable dense set of X.

Ex:- R is SS with usual metric ($\because$ Q in R is countable as well as dense)

Nowhere Dense set A subset A of MS X is said to be Nowhere Dense in X if $(\bar{A})^\circ = \phi$ i.e $\bar{A}$ contains no int. point.

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Ex: Any finite set is nowhere dense in R with usual metric.

21-9-23 Theorem:- Let (Y, d_Y) be a subspace of a ms (X,d) if Y is complete then it is closed.

Let x be a lt point of Y then $\exists$ a seq $\{x_n\}$ of distinct points of Y which converges to x. Since each conv. seq. is cauchy, it is a cauchy seq. in Y.
Also since Y is complete, the lt point x of this seq. must lie in Y.
Thus Y is closed.

Theorem:- Let (X,d) be a complete MS & (Y, d_Y) be a subspace of (X,d) then Y is complete iff it is closed.

If Y is complete subspace of (X,d) then by previous theorem it is closed.
Conversely, Assume that Y is closed subspace of complete MS (X,d).
Let $\{x_n\}$ be a cauchy seq. of points of Y
Since X is complete, this seq. converge to a point $x \in X$
$\because$ Y is closed & x being a lt point of the seq. $\{x_n\}$ & $\{x_n\}$ $x_n \in Y$
$\Rightarrow$ $x \in Y$
Thus each cauchy seq. of a point of Y converge to a point of Y.
Hence Y is complete.

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Cantor's Intersection Theorem:-

Let (X,d) be a compute MS & let seq $\{F_n\}$ be a decreasing seq. of non empty closed subsets of X,

$(F_{n+1} \subseteq F_n)$ $\forall n$

s.t $\delta(F_n) \longrightarrow 0$ as $n \longrightarrow \infty$

Then ~~Into~~ $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

$F_1 \supset F_2 \supset F_3 \supset F_4 \ldots \supset F_n \ldots$

$x_1 \in F_1$, $x_2 \in F_2$, $x_n \in F_n$

Pf:- construct a seq $\{x_n\}$ in X. x_n By seleting a point $\{x_n\} \in F_n$ for each $n \in \mathbb{N}$.

since $F_{n+1} \supseteq F_n$ $\forall n$.

we have $x_n \in F_n \subset F_m$ $\forall n>m$ → decr seq.

we claim that x_n is cauchy seq

Let $\epsilon > 0$, since $\delta(F_n) \to 0$ $\exists$ a +ve Integer N s.t $\delta(F_n) < \epsilon$, $\forall n > m$.

since F_n is a decreasing seq., we have $F_m, F_n \subseteq F_N$, $\forall n,m > N$

$\therefore$ $x_n, x_m \in F_N$ $\forall n,m > N$ & thus

$d(x_n, x_m) < \delta(F_N) < \epsilon$ $\forall n,m > N$

Hence seq $\{x_n\}$ is cauchy.

$\because$ X is complete $\exists$ $x \in X$ s.t $\{x_n\} \longrightarrow x$.

we claim that $x \in \bigcap_{n=1}^{\infty}(F_n)$

Let 'n' be fixed then the subsequence $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$ of $\{x_n\}$ is contained in F_n

Previous result (say)

f still converges to x

$\because$ every subsequence of conv. seq is conv.

but F_n being a closed subspace of complete MS (X, d), it is complete (previous thm)

$\therefore$ $x \in F_n$.

It is true for each $n \in N$.

Hence $x \in \bigcap_{n=1}^{\infty} F_n$ i.e $\bigcap_{n=1}^{\infty} F_n \neq \phi$

Finally to establish that x is the only point in $\bigcap_{n=1}^{\infty} F_n$

Let $y \in \bigcap_{n=1}^{\infty} F_n$ then x & y both are in F_n $\forall n$

$\therefore$ $0 \leq d(x,y) \leq \delta(F_n) \longrightarrow 0$ as $n \to \infty$

Thus $d(x,y) = 0$ or $d(x,y) \longrightarrow 0$

then $x = y$