$f(x) = \cos x$

5/July

BMC 101

Real Number System

① Completeness in the set of real numbers.
Every non empty set of real numbers which is bounded above has the supremum or the least upper bound $\mathbb{R}$.

0   1 $\longrightarrow$ upper bound $[1, \infty)$

<u>Upper bound</u> - A real number x is said to be an upper bound of a set A if $x \; \forall a \in A$ is such that $a \leq x$

<u>Least Upper bound</u> - The least upper bound or supremum of a set A is the smallest element of set of all upper bounds of A

<u>Lower bound</u> - A real number x is said to be a lower bound of a set B if $x \; \forall a \in B$ is such that $a \geq x$.

<u>Greatest Lower Bound</u> - The greatest lower bound or infimum of a set B is the greatest element of set of all lower bounds of B.

<u>Bounded set</u> - If a set of a real number is said to be bounded if there exists atleast one

$A \setminus B = A - B$

upper bound & lower bound.

Unbounded - If a set of R ~~exit~~ ~~is either~~ has no lower bound or no upper bound or both do not exist.

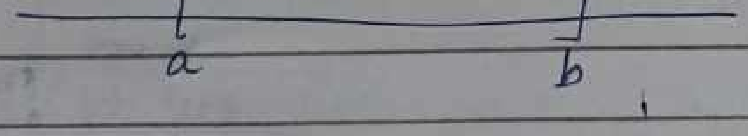
Bounded Above - A set of R is said to be bounded above if there exists atleast one upper bound.

Bounded Below - A set of R which has atleast one lower bound.

Ex -

	Set		Upper	Lower
1	$[a, b]$	bounded above + below	$[b, \infty)$	$(-\infty, a]$
2	(a, b)	~~un~~bounded	$[b, \infty)$	$(-\infty, a]$
3	$\mathbb{R}$	unbounded		
4	$\{1\}$	bounded	$[1, \infty)$	$(-\infty, 1]$
5	ϕ	bounded	R	R
6	$(1,2) \cup \{4\}$	bounded	$[4, \infty)$	$(-\infty, 1]$
7	$(1,2) \setminus (1.5, 1.7)$	bounded	$[2, \infty)$	$(-\infty, 1]$
8	$(-1, \infty)$	unbounded		$(-\infty, -1]$
9	$(-\infty, 0)$	unbounded	$[0, \infty)$	~~$[0, \infty)$~~
10	$(-\infty, 1) \cup (2, \infty)$	unbounded		

1 $[a, b]$



7 $(1, 1.5] \cup [1.7, 2)$

Let $l = a$ for a sequence $\{x_n\} = \{b\}$

Then : acc to the definition $|x_n - a| < \varepsilon \ \forall\ n \geq N$

~~At If~~ $x_n = b$: $|b - a| < \varepsilon \ \forall\ n \geq N$

when $b = a$

This is possible only when $b = a$

$\Rightarrow$ For a constant sequence, the constant value is the limit of the sequence

BMC 101 $\exists$: there exists

⇒ completeness
$(1,2)$ & $(-\infty, 0)$
⇒ axiom - cannot be proved - is true!
⇒ postulate
⇒ ~~metric~~
⇒ theorem

⇒ Density of rationals & irrationals
⇒ cardinality
⇒ If we want to define the cardinality of ∞ num, we need transfinite num.
⇒ countably infinite
⇒ $\aleph_0$ (~~Aliphnot~~ nought)
⇒ uncountably ~~finite~~ ~~infinite~~ (C)

⇒ The set of ~~rea~~ $\mathbb{Q}$ does not satisfy the completeness axiom of $\mathbb{R}$.
$\mathbb{Q}$ ranges b/w $(-\infty, \infty)$ & contains elements such as $\frac{1}{2}, 2, \frac{3}{2}$, etc. $\{-\infty < x < \infty \mid x = \frac{p}{q}$, where $p \in \mathbb{Z}$ & $q \neq 0$, $q \in \mathbb{N}\}$
$\mathbb{Q} \in \frac{p}{q}$

$\mathbb{Z}$ - $\mathbb{I}$
Let $a \in \mathbb{R}$ is an upper bound of $\mathbb{Q}$ then ~~the~~ for each element, $x \in \mathbb{Q}$ is such that $x \leq a$
Now let $a \in \mathbb{Q}$ $\exists$ $p \in I$ & $q \in \mathbb{N} \Rightarrow a = \frac{p}{q}$.
& we also know that we can define a rational no. x such that $x > a$
It is a contradiction to our assumption that a is an upper bound.

When $a \in Q^c$, $a \neq \frac{p}{q}$

& there will always exist a $\mathbb{R}$ x which is greater than a ⇒ it is a contradiction to our assumption
Hence, $\mathbb{Q}$ does not have any upper bound

⇒ Archimedean Property of Real Number

→ If a & b be any 2 +ve real numbers then there exists a +ve integer or a natural number n such that $na > b$

$a, b \in \mathbb{R}$ $\exists$ $n \in \mathbb{N}$ ∋: $na > b$ or $\exists$ $n \in \mathbb{N}$ ∋: $nb > a$

→ For any +ve $\mathbb{R}$ r $\exists$ a $\mathbb{N}$ ∋: $\frac{1}{n} < r$

$a > nb$
$a = 1$, $b = r$
$\frac{1}{r} > n$
$\Rightarrow \frac{1}{n} > r$

⇒ Corollary:

1. If 'a' be a +ve $\mathbb{R}$ & 'b' any $\mathbb{R}$ then $\exists$ a +ve $\mathbb{Z}$ n such that $na > b$
2. For any +ve real number 'a' $\exists$ a +ve $\mathbb{Z}$ n such that $n > a$
3. If 'a' be any $\mathbb{R}$ then $\exists$ a +ve $\mathbb{Z}$ n such that $n > a$

→ Proof: Let a, b be 2 +ve $\mathbb{R}$
Let us suppose if possible that for all +ve integers $\mathbb{N}$ $na < b$
Thus, the set $S = \{na : n \in \mathbb{Z}^+\}$ is bounded above, b being an upper bound. By the completeness property of $\mathbb{R}$ this set S must have the sup. say M.

To prove corollary $\rightarrow$ prove theorem then write similarly.

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$\therefore na \leq M \ \forall n \in Z^+$

$\Rightarrow (n+1)a \leq M \ \forall n \in Z^+$

$\Rightarrow na \leq M-a \ \forall n \in Z^+$

That is $M-a$ is an upper bound of S. Thus, a number $M-a$ less than the sup. (least upper bound) is an upper bound of S, which is a contradiction hence, our assumption is wrong.

$\forall$ = for all
$\ni$: Such that :

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BMC 101

⇒ Let $\alpha \in \mathbb{Q}$
$n \in$ prime
$\alpha\sqrt{n} \in \mathbb{Q}^c$

⇒ Density of $\mathbb{Q}$ & $\mathbb{Q}^c$ in $\mathbb{R}$:

• Density of $\mathbb{Q}$ in $\mathbb{R}$: If $x \in \mathbb{R}$ & $y \in \mathbb{R}$ with $x<y$ then $\exists$ an $z \in \mathbb{Q}$ such that $x<z<y$.

Proof ① First we suppose that x is +ve ⇒ y is also +ve
$0<x<y$ & $y-x>0$.

∴ Consider $\frac{1}{(y-x)}$ so that this is also +ve.

Hence, by Archimedean property there exists a $\mathbb{N}$ n such that $n > \frac{1}{y-x}$

Multiplying by $(y-x)$ & simplifying we get
$n(y-x) > 1$

Or we can say that $nx+1 < ny$

Since nx is +ve hence, $\exists$ a $\mathbb{N}$ $m \ni: (m-1) \le nx < m$ (*)

Now the first inequality in (*) ⇒ $m \le nx+1$
& we have already shown that $nx+1 < ny$, this means that $m<ny$.

However, the 2nd inequality in (*) ⇒ $nx<m$

Combining we have : $nx < m < ny$

OR equivalently:

$$x < \frac{m}{n} < y$$

Here, $\frac{m}{n}$ is the required $\mathbb{Q}$ i.e. $z = \frac{m}{n}$ ⟶ (i)

A Corollary
Density of $\mathbb{Q}^c$ in $\mathbb{R}$: If $x \in \mathbb{R}$ & $y \in \mathbb{R}$ & $x<y$, then $\exists$ an $\mathbb{Q}^c$ $z \in \mathbb{Q}^c \ni: x<z<y$.

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(2) On the other hand, suppose that $x \le 0$. Choose an $\mathbb{Z}$ k such that $k > |x|$

Now if we consider $x+k$ & $y+k$ which are both +ve, then we can use the prev. argument to conclude that there exists a $\mathbb{Q}$ q with $x+k < q < y+k$

Hence, $\Rightarrow x < q-k < y$

It is obvious that $q-k \in \mathbb{Q}$ & $q-k = z$

(ii) ~~proof~~:

If we consider $\frac{x}{\sqrt{2}}$ & $\frac{y}{\sqrt{2}}$ then by prev. theorem

there exists a $\mathbb{Q}$ $w \in \mathbb{Q}$ $\ni: \frac{x}{\sqrt{2}} < w < \frac{y}{\sqrt{2}}$

$\Rightarrow x < \sqrt{2}\, w < y$

Hence, $w\sqrt{2}$ is the required irrational no which $= z$.

* $\boxed{\text{Directional Derivative}}$ → (particular case) Differentiation

$\boxed{\text{SEQUENCE}}$

⇒ A sequence of $\mathbb{R}$ is a function from set of $\mathbb{N}$ → set of $\mathbb{R}$

$f: \mathbb{N} \to \mathbb{R}$

$\boxed{\text{ex:}}$ $a_1 = f(1) = 1$

$a_2 = f(2) = 2$

$a_3 = f(3) = 3$

$\vdots$

$a_n = f(n) = n$

→ Sequence notation $(a_n)_{n\in\mathbb{N}}$ OR $\{a_n\}_{n\in\mathbb{N}}$

ex. ① $(a_n) = n$ ③ $(a_n) = \frac{1}{n}$ ④ $(a_n) = (-1)^n$ ⑤ $(a_n) = \left(1+\frac{1}{n}\right)^n$

② $(a_n) = 1$

⑥ $(a_n) = (\sin n)$

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* Bounded Sequence - A sequence is said to be bounded if $\exists$ a +ve $\mathbb{R}$: M ?: $|a_n| < M \ \forall \ n \in \mathbb{N}$

Ex - $(a_n) = n$ → Unbounded. ∴ M does not exists

$\boxed{\text{write st.}}$

* A sequence which is not bounded, then it will be unbounded
* A sequence is either bounded above or bounded below or both. or can also be neither.

* Neither bounded below nor bounded above:

$$\boxed{(a_n) = \begin{cases} -n, & n \text{ is odd} \\ n, & n \text{ is even} \end{cases}}$$

* $(a_n) = \left(\frac{1}{n}\right)$

Bounded.

$$\boxed{|a_n| < M = 1 \text{ 1 or any } \mathbb{N} > 1}$$

$\boxed{e = 2.73}$ approx

* $(a_n) = (e^n)$ unbounded + bounded below

Lower bound = $\boxed{e^1 - 1}$

* $(a_n) = \left(\frac{n}{1+2n}\right)$: Bounded

* $(a_n) = (\sin n)$

$M = 1.1$

BMC 101

→ Limit of a Sequence

A real number l is said to be a limit of a sequence $\{x_n\}$ if for every $\varepsilon > 0$ $\exists$ a +ve $\mathbb{Z}$, $N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon$, $\forall n \geq N$, $n \in \mathbb{N}$

$(x_n) \to$ sequence
$l \to$ limit
$|x_n - l| < \varepsilon$
$\varepsilon > 0$

$|x_n - l| \in \mathbb{R}$ must be small from every +ve number
$(x_n - l) \underset{\text{converges to}}{\to} 0 \Rightarrow (x_n - l) = 0$

$$\boxed{\lim_{n \to \infty} (x_n) = l}$$

l is the limit of x_n.

Example: ① $\{x_n\} = \{a\}$ constant seq. + real seq. $\} \forall \varepsilon > 0\ \exists N \in \mathbb{N} \ni |x_n - a| < \varepsilon$, $\forall n \geq N$ $\}$ let $l = a$, $n = 1 \Rightarrow N = 1$

② $\{x_n\} = \begin{cases} a & \text{if } n = 1 \text{ to } 4 \\ b & \text{if } n \geq 4 \end{cases}$ eventually constant seq.

$|x_n - a| \to 0$.
$l = a \to$ limit of seq.

③ $\{x_n\} = \{0\} \Rightarrow$ limit $= 0$
let $l = 1$
By definition $\forall \varepsilon > 0$, $\exists N \in \mathbb{N} \mid |x_n - l| < \varepsilon$, $\forall n \geq N$
$|0 - 1| < \varepsilon$, $\forall n \geq N$
$|-1| < \varepsilon$, $\forall n \geq N$
$1 < \varepsilon$, $\forall n \geq N$
But if $\varepsilon = 1/2$ then $\Rightarrow l \neq 0$ due to the contradiction toward our ~~assu~~ assumption.

→ Prove that the limit of a sequence is unique. ∘ contradiction

$\boxed{\text{Take } \varepsilon = c}$

② $x_n = \begin{cases} a & \text{if } 1 \le n \le 4 \\ b & \text{if } n > 4 \end{cases}$

$x_1 = x_2 = x_3 = x_4 = a$

$x_5 = x_6 = \dots = x_n = b$

Let $l = 3$ be a limit of seq. (x_n)

for each $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon$ $\forall$ $n \ge N$

$\lvert x_n - 3\rvert < \varepsilon$. $\forall$ $n \ge N$	$\lvert x_n - 3\rvert < \varepsilon$ $\forall$ $n \ge N$
If $a \ne 3$ $\lvert x_n - 3\rvert \ne 0$	If $b \ne 3$ $\lvert x_n - 3\rvert \ne 0$

∴ $l = 3$ is not the limit

Let $l = a$ be a limit of seq. (x_n).

for each $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - a| < \varepsilon$ $\forall$ $n \ge N$

Let $N = 1$ $|x_n - a| < \varepsilon$ → $|a - a| < \varepsilon$

Inequality satisfies

Let $N = 5$ $|x_n - a| < \varepsilon$ → $|b - a| < \varepsilon$

Inequality does not satisfy

Let $l = b$ be the limit of seq (x_n)

for each $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - b| < \varepsilon$ $\forall$ $n \ge N$

Let $N = 1$ does not satisfy

Let $N = 5$ satisfies ✓

Let $N = 6$ satisfies ✓

$l = b$ is the limit for $n \ge 5$ → $N = 5$

Note: Limit of any sequence is always unique

We need max N for which every $n \ge N$ has the constant limit.

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Proof

Let b_1 & b_2 such $b_1 \neq b_2$ are the limit of a seq. $\{x_n\}$ $\forall \epsilon > 0$ $\exists N_1 \in \mathbb{N}$

$\ni: |x_n - b_1| < \epsilon$, $\forall n \geq N_1$ — ①

& $\forall \epsilon > 0$, $\exists N_2 \in \mathbb{N}$

$\ni: |x_n - b_2| < \epsilon$, $\forall n \geq N_2$ — ②

Let N is $\max\{N_1, N_2\}$ then ① & ② are satisfied for all $n \geq N$

Now, if we take any $\mathbb{N}$ ~~#~~ $n \geq N$ we have

$$|b_1 - b_2| \cancel{\#} = \cancel{|x_n - b_2| + |x_n - b_1|}$$

$$= |x_n - x_n + b_1 - b_2|$$

$$= |(x_n - b_2) - (x_n - b_1)|$$

$$\boxed{|a_1 \pm a_2| \leq |a_1| + |a_2|}$$

$$\leq |x_n - b_2| + |x_n - b_1| = 0$$

$$b_1 = b_2$$

contradiction!

$\therefore$ A sequence will not have more than 1 limit.

Our assumption is wrong. ~~so~~ all sequences will have a a unique limit

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* Limit points can be more than 1, limit will be unique

① $\{x_n\} = \left\{\frac{1}{n}\right\} \longrightarrow 0$ (convergent)

We claim that $l = 0$

for each $\varepsilon > 0$ $\exists$ a $N \in \mathbb{N}$: $|x_n - l| < \varepsilon, \forall\ n \geq N$

$\Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon, \forall\ n \geq N$

Now, we have $\frac{1}{n} < \varepsilon$ & $\frac{1}{\varepsilon} < n$

N is that smallest $\mathbb{N}$ which is greater than $\frac{1}{\varepsilon}$.

Hence, 0 is the limit of the sequence $\{x_n\}$

- If $\varepsilon = 1 \Rightarrow \frac{1}{1}$; $\Rightarrow$ ~~$n_0 = 2$~~ $N = 2$

$$|x_n - l| < 1$$
$$\left|\frac{1}{n} - 0\right| < 1 \quad \forall\ n \geq 2$$

- If $\varepsilon = 0.1 \Rightarrow \frac{10}{1} \Rightarrow$ ~~$n_0 =$~~ $N = 11$

$$|x_n - l| < 0.1 = \frac{1}{10}$$

$$\left|\frac{1}{n} - 0\right| < \frac{1}{10} \quad \forall\ n \geq 11$$

- N can be same or diff for every ε

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$\{x_n\} = \{\frac{1}{n}\}$

- Let $l = 1$:

We claim that ~~$\frac{1}{n}$~~ the sequence has limit $l = 1$

By definition for each $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N} : |x_n - l| < \varepsilon$, $\forall\ n \geq n_0$

$\Rightarrow \left|\frac{1}{n} - 1\right| < \varepsilon,\ \forall\ n \geq n_0$

If we take $\varepsilon = 1/2$

$\left|\frac{1}{n} - 1\right| < \frac{1}{2},\ \forall\ n \geq n_0$

Since the definition cannot be applicable

Hence, $l = 1$ is not the limit

* Subsequence: A subsequence of a sequence is also a sequence which is defined from an infinite subset of set of $\mathbb{N} \longrightarrow$ set of $\mathbb{R}$

subsets: $2\mathbb{N}$

$3\mathbb{N}$

$2^n \cdots$

$(2, \infty)$

$\boxed{f : 2\mathbb{N} \longrightarrow \mathbb{R}} \rightarrow$ R: is subset of $\boxed{f : \mathbb{N} \longrightarrow \mathbb{R}}$

- $(x_n) = (-1)^n$.

$(x_n) = \begin{cases} 1 & n \text{ is even or } n = 2m \\ -1 & n \text{ is odd or } n = 2m - 1 \end{cases} \qquad m \in \mathbb{N}$

- $x : 2\mathbb{N} \rightarrow \mathbb{R}$

$\{x_{2m}\} = \{1\}$ + $\{x_{2m-1}\} = \{-1\}$

$x : 2\mathbb{N} - 1 \longrightarrow \mathbb{R}$

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→ <u>Limit point of a sequence</u>: If a number l is said to be a limit point of a sequence $\{x_n\}$ then there exists a subsequence $\{x_{n_k}\}$ which has limit l.

Ex- $\{x_n\} = (-1)^n$ → this has 2 limit points but no limit

$\{x_{2m}\} = \{1\} \longrightarrow 1$

$\{x_{2m-1}\} = \{-1\} \rightarrow -1$

→ Sequence has limit point which is the limit of subsequence.

$\boxed{\text{Note}}$ If $\exists$ 2 subsequences of a sequence which have different limit points then the limit of the sequence will not exist but sequence will have 2 diff. limit points.

$$\{x_n\} = \begin{cases} 1 & \text{when } n = 3k, \; k \in \mathbb{N} \\ 3 & \text{when } n = 5k. \\ 5 & \text{when } n = 7k \end{cases}$$

~~Ex = 15 = 3k₁ = 5k₂~~ 6 when $3k_1 = 5k_2$ or $3k_3 = 7k_4$ or $5k_5 = 7k_6$

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Result (i): A sequence may have more than one limit point but the limit (if exists) will be unique.

(ii) If a sequence has limit 'l' then sequence will have only one limit point i.e. 'l'.

"A sequence is said to be convergent if it has a limit".

Ex - $\lim_{n\to\infty}\left(\frac{3+2\sqrt{n}}{\sqrt{n}}\right)$

$\Rightarrow \{x_n\} = \frac{3+2\sqrt{n}}{\sqrt{n}}$

$\{x_1\} = 5$

$\{x_\infty\} = \frac{\infty}{\infty}$, so we have to apply L-hopital rule

$\therefore \lim_{n\to\infty}\left(\frac{3+2\sqrt{n}}{\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{3}{\sqrt{n}}+2\right) = \lim_{n\to\infty}(2) = 2$ **Ans**

For exams:- If a similar ques comes then do by definition.

Sol^n: Let $l=2$ be the limit of sequence $\{x_n\} = \left\{\frac{3+2\sqrt{n}}{\sqrt{n}}\right\}$

then $\forall\ \varepsilon>0\ \exists\ n_0 \in \mathbb{N}$ such that

$$\left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| < \varepsilon,\ \forall\ n \geq n_0$$

$$\left|\frac{3+2\sqrt{n}-2\sqrt{n}}{\sqrt{n}}\right| < \varepsilon,\ \forall\ n \geq n_0$$

$$\left|\frac{3}{\sqrt{n}}\right| < \varepsilon,\ \forall\ n \geq n_0$$

Squaring both the sides $= \frac{9}{n} < \varepsilon^2,\ \forall\ n \geq n_0$

$$\frac{9}{\epsilon^2} < n, \forall n \geq n_0$$

n_0 is the smallest $\mathbb{N}$ which is greater than $\frac{9}{\epsilon^2}$

Ex - Prove that the sequence $\left\{\frac{2n-3}{n+1}\right\}$ is convergent

Solⁿ: $\lim_{n\to\infty} \left\{\frac{2n-3}{n+1}\right\} = \lim_{n\to\infty} \left\{\frac{2-\frac{3}{n}}{1+\frac{1}{n}}\right\} = 2$

Let $l = 2$ be limit of sequence $\left\{\frac{2n-3}{n+1}\right\}$ then $\forall \epsilon > 0$

$\exists n_0 \in \mathbb{N}$ such that

$$\left|\frac{2n-3}{n+1} - 2\right| < \epsilon, \forall n \geq n_0$$

$$\left|\frac{2n-3-2n-2}{n+1}\right| < \epsilon, \forall n \geq n_0$$

$$\left|\frac{-5}{n+1}\right| < \epsilon, \forall n \geq n_0$$

$$\frac{5}{\epsilon} < n+1, \forall n \geq n_0$$

$$\frac{5}{\epsilon} - 1 < n, \forall n \geq n_0$$

n_0 is the smallest natural number which is greater than $\frac{5}{\epsilon} - 1$.

Ex - $\{x_n\} = \{\sin 1/n\}$

Solⁿ: $\lim_{n\to\infty} \{\sin 1/n\} = \sin(1/\infty) = \sin 0 = 0$

Let $l = 0$ be the limit of $\{x_n\} = \{\sin 1/n\}$, then

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$\forall\ \epsilon > 0,\ \exists\ n \in \mathbb{N}$ such that

$$\left|\sin\left(\frac{1}{n}\right) - 0\right| < \epsilon \quad \forall\ n \geq n_0$$

$\left|\sin \frac{1}{n}\right| < \epsilon \longrightarrow$ Putting value of $n \in \mathbb{N}$ then limit will be in b/w $[0,1]$.

$\therefore \quad |\sin 1/n| < |1/n| < \epsilon$

$$\boxed{n > \frac{1}{\epsilon}}$$

Ex - $\{S_n\} = \left\{1 + \frac{(-1)^n}{n}\right\}$

Sol^n $\lim_{n\to\infty} \left\{1 + \frac{(-1)^n}{n}\right\} = 1$

Let $l = 1$ be the limit of $\{S_n\} = \left\{1 + \frac{(-1)^n}{n}\right\}$. then $\forall$

$\epsilon > 0\ \exists\ n \in \mathbb{N}$ such that $\left|1 + \frac{(-1)^n}{n} - 1\right| < \epsilon \ \forall\ n \geq n_0$

$$\left|\frac{(-1)^n}{n}\right| < \epsilon \ \forall\ n \geq n_0$$

$$\frac{1}{n} < \epsilon \ \forall\ n \geq n_0$$

$$\frac{1}{\epsilon} < n \ \forall\ n \geq n_0$$

n_0 is the smallest $\mathbb{N}$ which is greater than $1/\epsilon$.

Theorem - Every convergent sequence is bounded.

Proof: Let a sequence converge to limit 'l' then by definition of limit $\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that

$$|x_n - l| < \epsilon,\ \forall\ n \geq n_0$$

$\Rightarrow \boxed{l-\varepsilon \leq x_n \leq l+\varepsilon} \ \forall \ n \geq n_0$

Suppose: $g = \min\{l-\varepsilon, x_1, x_2, \ldots, x_{n_0-1}\}$
$G = \max\{l+\varepsilon, x_1, x_2, \ldots, x_{n_0-1}\}$

Hence, $g < x_n < G \ \forall n \in \mathbb{N} \ \& \ n \geq n_0$

<u>Example</u>: $\{x_n\} = 1, 2, 3, 4, 4, 4, 4, \ldots$

$l = 4 \ \& \ n_0 = 4$
$|x_n - 4| < \varepsilon \ \forall \ n \geq 4$

$-\varepsilon < (x_n - 4) < \varepsilon \ \forall \ n \geq 4$
$4 - \varepsilon < x_n < \varepsilon + 4 \ \forall \ n \geq 4$
$g = \min\{4-\varepsilon, 1, 2, 3\}$
$G = \max\{4+\varepsilon, 1, 2, 3\}$
For $\varepsilon = 1$, $g = 1$ which is lower bound
$G = 5$ which is upper bound.

$\xi_1(x_0)$

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Theorem - Every convergent sequence is bounded & has a unique limit.

(Prove that prev. 2 theorems)

Limit Point of a sequence: A no. ξ_1 is said to be a limit point of a sequence $\{x_n\}$ if every neighbourhood of ξ_1 contains an infinite no of members of sequence.

Neighbourhood - $(\xi_1 - \varepsilon, \xi_1 + \varepsilon)$

OR

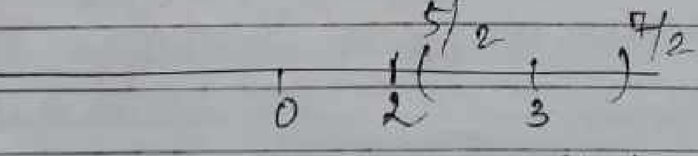
ξ_1 is a limit point of a sequence if for any +ve $\mathbb{R}$ $\varepsilon > 0$ $\{x_n\} \in (\xi_1 - \varepsilon, \xi_1 + \varepsilon)$ for an infinite no. of values of 'n', i.e., $|x_n - \xi_1| < \varepsilon$ for infinitely many values of 'n'.

Ex $\{x_n\} = \begin{cases} 1, & n \text{ is even} \\ 2, & n \text{ is odd} \end{cases}$

① $\xi_1 = 3$ for example (limit point assumption)

$\forall \varepsilon > 0$ we need to make a neighbourhood of ξ_1

$\varepsilon = 1/2$



No occurence of members of $\{x_n\}$.

$\Rightarrow \xi_1 = 3$ is not a limit point of the sequence

② $\xi_1 = 1$ (limit point assumption)

$\Rightarrow \xi_1 = 1$ is a limit point of the sequence.