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$\Rightarrow f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{(x-2)}$ + L hopital rule

$\lim_{x \to 2} f(x) = l = 4$

★→ If we say $\lim_{x \to a} f(x) = l$ then $\forall \varepsilon > 0 \; \exists \; \delta > 0$: $|f(x) - l| < \varepsilon$ whenever $|x-a| < \delta$ | $l \in \mathbb{R}$

To prove: $f(x) = \dfrac{x^2-4}{x-2}$

$\lim_{x \to 2} f(x) = ?\ (l) = 4$

Proof: $\forall \varepsilon > 0 \; \exists \; \delta > 0$: $|f(x) - l| < \varepsilon$ whenever $|x-a| < \delta$

$$\Rightarrow \left| \frac{x^2-4}{x-2} - 4 \right| < \varepsilon \text{ whenever } |x-2| < \delta$$

$$\Rightarrow \left| \frac{x^2-4-4x+8}{x-2} \right| < \varepsilon \text{ whenever } |x-2| < \delta$$

$$\Rightarrow \left| \frac{x^2-4x+4}{x-2} \right| < \varepsilon \text{ whenever } |x-2| < \delta$$

$$\Rightarrow \left| \frac{(x-2)^2}{(x-2)} \right| < \varepsilon \text{ whenever } |x-2| < \delta$$

$$\Rightarrow |(x-2)| < \varepsilon \text{ whenever } |x-2| < \delta$$

on comparing : $\boxed{\varepsilon = \delta}$

Hence, $L = 4$ is the limit of the given function

→ Find $\lim_{x \to 1} \frac{x^2-4}{x-2} = 3$ = limiting value

⇒ Find $\lim_{x \to 2} (x^2-4)/(x-2) = 4$

LHL: $\lim_{h \to 0} \cancel{\frac{f(x-h) - f(x)}{-h}}$

$\cancel{\lim_{h \to 0} \frac{(x-h)^2 - 4 - (x^2-4)}{-h}}$

$\cancel{\lim_{h \to 0} \frac{x^2+h^2-2xh-4}{x-h-2}}$

LHL: $\lim_{h \to 0} f(a-h)$

$\lim_{h \to 0} \frac{(2-h)^2-4}{(2-h)-2}$

$\lim_{h \to 0} \frac{\cancel{4}+h^2-4h-\cancel{4}}{-h}$

$\lim_{h \to 0} \frac{h-4}{-1}$

$= 4$

RHL: $\lim_{h \to 0} \cancel{\frac{f(x+h) - f(x)}{h}}$

$\cancel{\lim_{h \to 0} \frac{(x+h)^2-4}{(x+h)-2}}$

$\cancel{\lim_{h \to 0} \frac{x^2+h^2+2xh-4}{x+h-2}}$

RHL:

$\lim_{h \to 0} f(a+h)$

$\lim_{h \to 0} \frac{(2+h)^2-4}{(2+h)-2}$

$\lim_{h \to 0} \frac{\cancel{4}+h^2+4h-\cancel{4}}{h}$

$\lim_{h \to 0} h+4$

$= 4$

⇒ $f(x) = |x|$ at $x \to 0$: $\lim_{x \to 0} |x| = 0$

$a = 0$

LHL: $\lim_{h \to 0} f(a-h)$

RHL: $\lim_{h \to 0} f(a+h)$

LHL: $\lim_{h \to 0} |0-h|$	RHL: $\lim_{h \to 0} |0+h|$
$\lim_{h \to 0} h$	$\lim_{h \to 0} h$
$= 0$	$= 0$

$\Rightarrow f(x) = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$

$x \longrightarrow 0 \Rightarrow a = 0$

Sol$^n \rightarrow$

For $x \geq 0$	For $x < 0$
RHL: $\lim_{h \to 0} f(a+h)$	LHL: $\lim_{h \to 0} f(a-h)$
$\lim_{h \to 0} 1$	$\lim_{h \to 0} -1$

LHL $\neq$ RHL

Limit does not exist

Continuity

☆ A function is said to be continous at 'a' if

$$\lim_{x \to a} f(x) = f(a)$$

✱ If a function is continuous in $\mathbb{R}$ then it is continuous at every point/value $\in \mathbb{R}$.

Ex $- f(x) = |x| \longrightarrow$ continuous

$\rightarrow$ $f(x) = \begin{cases} \dfrac{\sin x}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ on $\mathbb{R}$

$\rightarrow$ Removable Discontinuity

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

L.H. Rule : $\lim_{x \to 0} \dfrac{\cos x}{1} = \lim_{x \to 0} \cos 0 = 1$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}{x}$$

$$=\lim_{x\to 0} 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots$$

$$=1$$

$\Rightarrow \lim_{x\to 0}\frac{\sin x}{x}=1$

$f(0)=0$

discontinuous at 0.

Let $a\in\mathbb{R} : a\neq 0$

$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{\sin x}{x}=\frac{\sin a}{a}=f(a)$

Continuous for : $\mathbb{R}-\{0\}$

olve

→ ① $\lim_{x\to 0}\frac{\cos x}{x}$ = not defined

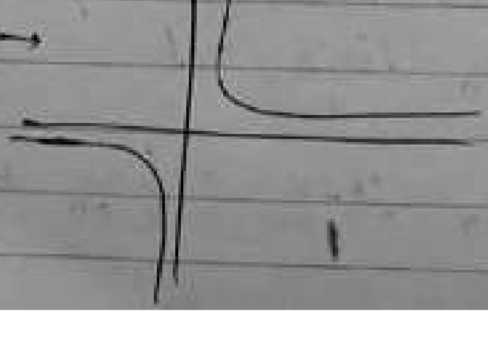
→ ② $\lim_{x\to\infty}\sin x$ = not defined

→ ③ $\lim_{x\to 0} x\sin\frac{1}{x}=0$

① $\lim_{x\to 0} f(0-h)=\frac{\cos(0-h)}{-h}=\frac{\cos(h)}{-h}=\frac{-\cos h}{h}=-\infty$

$\lim_{x\to 0} f(0+h)=\frac{\cos(0+h)}{h}=\frac{\cos h}{h}=\frac{1}{0}=\infty$

① $\lim_{x\to 0}\frac{1}{x}$ graph →



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* Closure: $cl(A) = A \cup D(A)$

* A is closed if $cl(A) = A$
$\Rightarrow D(A) = A$

$\exists N_\varepsilon(x)$ s.t. ~~$x \in$~~ $N_\varepsilon(x) \subset A$

$\Rightarrow \mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \cdots$

* Countable union of open set is open
→ $\mathbb{N}$ is closed

* $\mathbb{Z}$
→ ~~$\mathbb{R} - \mathbb{Z}$~~ $\mathbb{Z} = \{-n, \ldots, -1, 0, 1, \ldots, n\}$
→ $\mathbb{Q} \Rightarrow D(\mathbb{Q}) = \mathbb{R}$
$y \in N_\varepsilon(y) \subset \mathbb{Q}$ } neither open, nor closed.

$\Rightarrow \mathbb{Q}^c$ – neither open, nor closed
$\Rightarrow D(\mathbb{Q}^c) = \mathbb{R}$
$\Rightarrow y \in N_\varepsilon(y) \subset \mathbb{Q}$

Closed sets
1. Finite Set
2. $\mathbb{N}$
3. $\mathbb{Z}$

Neither open nor closed
1. $\mathbb{Q}$, $\mathbb{Q}^c$

Open + Closed
1. $\mathbb{R}$
2. ϕ

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	Set	Seq	Function
Limit point	✓	✓	
Limit		✓	✓

→ $f: A \to \mathbb{R} \ (A \subseteq \mathbb{R})$

* Isolated point & Singleton point

→ $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

Theorem - Let f & g be 2 continuous func^n

① $f \pm g$, αf are continuous

② $\frac{f}{g}$ is continuous $(g(x) \neq 0)$

③ $f \circ g$ is continuous

→ ① $\sin(x^2)$

② $\cos(2x)$

③ $\sin x + \tan x$

④ $|f|$ is also continuous

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① $\lim_{n \to \infty} \left(\frac{n^3}{n^2+n+1} \right)$

L-H Rule → $\frac{3n^2}{2n+1}$

L-H Rule → $\frac{6n}{2} = \infty$

~~L-H Rule → $\frac{6}{2} = 3$~~

② $\lim_{n \to 0} \left(\frac{\sin n}{n} \right)$

L-H Rule → $\frac{\cos n}{1} = 1$.

* limit exists → sequence is convergent.

③ $f(x) = \sin(e^x)$

④ $g(x) = \cos x + |\sin x| + |x-2|$

⑤ $h(x) = \frac{x^3 + x^4 + |x| + \sin x}{x^2+1}$

3.) ~~$\sin x$ lim~~ $f(x) = \sin x$

LHL: $\lim_{h \to 0} \sin(a-h)$	RHL: $\lim_{h \to 0} \sin(a+h)$
$= \lim_{h \to 0} \sin a \cos h - \cos a \sin h$	$= \lim_{h \to 0} \sin a \cos h + \cos a \sin h$
$= \lim_{h \to 0} \sin a \times 1 - 0$	$= \lim_{h \to 0} \sin a \times 1 - 0$
$= \lim_{h \to 0} \sin a$	$= \lim_{h \to 0} \sin a$

LHL = RHL = $f(a)$

continuous

$4n^2+2$ Page:

$g(x) = e^x$

LHL: $\lim_{h\to 0} g(a-h)$

$= \lim_{h\to 0} e^{(a-h)}$

$= \lim_{h\to 0} \frac{e^a}{e^h}$

$= e^a$

RHL: $\lim_{h\to 0} g(a+h)$

$\lim_{h\to 0} e^{(a+h)}$

$\lim_{h\to 0} e^a \times e^h$

e^a

$g(a) = LHL = RHL$

fog is continuous

4.) $A(x) = \cos x$

LHL: $\lim_{h\to 0} A(a-h)$

$= \lim_{h\to 0} \cos(a-h)$

$= \cancel{\lim_{h\to 0} \cos a \sin h + \cos h \sin a}$

$= 0 + \sin a$

RHL: $\lim_{h\to 0} A(a+h)$

$= \lim_{h\to 0} \cos(a+h)$

$= \cancel{\lim_{h\to 0} \cos a \sin h + \cos h \sin a}$

$= 0 - \sin a$

check:

(6) $a_n = \frac{(-1)^n}{n^2+1}$ → convergent, oscillatory

$\{a_n\} = \{ \frac{-1}{2}, \frac{1}{5}, \frac{-1}{10}, \ldots, \frac{(-1)^n}{n^2+1} \}$

(7) $a_n = \begin{cases} 1/n, & n = 2k \\ n, & n = 2k+1 \end{cases}$

6.) $|a_n| = \frac{1}{n^2+1}$

$\lim_{n\to\infty} \frac{1}{n^2+1} = \frac{1}{\infty} = 0$

$|a_n| \to 0$

$0 \leq |a_n| \leq \frac{1}{2}$

check for $2n$ & $2n+1$

(8) $a_n = n^2$ (9) $a_n = \frac{1}{n}$

$$a_n = \begin{cases} 1/n, & n = 2k \\ n, & n = 2k+1 \end{cases}$$

$$a_{2k} = \frac{1}{2k} \longrightarrow 0$$

$$a_{2k+1} = 2k+1 \longrightarrow \infty$$

★ Agar seq unbounded h phir bhi limit point exist kar sakta h.

Check continuous:

(10) $$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 2x-1, & x \in \mathbb{Q}^c \end{cases}$$

(11) $$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{Q}^c \end{cases}$$

(12) $$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Extended function

$$F(x) = \begin{cases} f(x), & x \neq 0 \\ \underset{\text{limit}}{l}, & x = 0 \end{cases} = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

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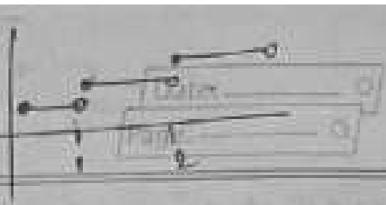

Continuity on an interval

Que ① Discuss the continuity of $f(x)$

$$f(x) = x - [x], \text{ at } x = 0$$

② $f(x) = \begin{cases} \dfrac{x e^{1/x}}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

1) LHL: $\lim_{h \to 0} f(a-h)$

$= \lim_{h \to 0} (a-h) - [a-h]$

$= \lim_{h \to 0} a - h - (a-1)$

$= \lim_{h \to 0} \cancel{a} - h - \cancel{a} + 1$

$= \lim_{h \to 0} 1$

RHL: $\lim_{h \to 0} f(a+h)$

$= \lim_{h \to 0} (a+h) - [a+h]$

$= \lim_{h \to 0} a + h - (a+1)$

$= \lim_{h \to 0} \cancel{a} + h - \cancel{a} - 1$

$= -1$

LHL $= \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} -h - [-h], \; h < 0$

$= \lim_{h \to 0} -h - (-1)$

$= \lim_{h \to 0} -h + 1 = 1$

RHL: $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} h - [h], \; h > 0$

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$= \lim_{h \to 0} h - 0 = 0$

LHL ≠ RHL

2) LHL: $\cancel{\lim_{h \to 0} f(a-h)}$

$\cancel{= \lim_{h \to 0} \frac{(a-h) e^{1/(a-h)}}{1 + e^{1/(a-h)}}}$

$\lim_{x \to 0} \frac{x e^{1/x}}{1 + e^{1/x}} \quad \cancel{\neq \frac{0}{1}}$

① solve normally
② divide by $e^{1/x}$

$= \lim_{x \to 0} \frac{x}{1 + e^{-1/x}} = \frac{0}{1 + \frac{1}{e^{1/0}}} = \frac{0}{1 + \frac{1}{\infty}} = \frac{0}{1+0}$

$= 0$

LHL: $\lim_{h \to 0} f(0-h) = f(-h)$

③ $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

LHL: $\lim_{x \to 0} f(a-h)$

$= \lim_{x \to 0} \frac{\sin 2(a-h)}{(a-h)}$

$= \lim_{x \to 0} \frac{\sin 2(-h)}{-h}$

$= \lim_{h \to 0}$

① **Dirichlet func^n** → f

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c \\ -1, & x \in \mathbb{Q} \end{cases}$$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

$a_n = [\mathbb{Q}]$
$a_n = [\mathbb{Q}^c] \to$ Irrational
$a_n \to a$
$f(a_n) = -1$
$f(a) = 1$

Theorem: A func^n f defined on an interval I is continuous at a point $c \in I$ iff for every sequence $[c_n]$ in I converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$

→ Signum func^n

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad \text{at } x = 0$$

$\dfrac{x}{|x|}, x \neq 0$

Let $a_n = \dfrac{(-1)^n}{n} \to 0 = 0$

$f(a_n) \to f(a) = 0$

$f(a_n) = (-1)^n \not\to f(a)$
$f(0) = 0$

⇒ $f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c \\ -1, & x \in \mathbb{Q} \end{cases}$

⇒ $f(x) = x^2$, at $x = 0$
$a_n = \frac{1}{n} \to 0$, $f(a_n) = 0$

$f(a_n) = \frac{1}{n^2} \longrightarrow 0$

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- **Theorem** - A function defined on an interval I is continuous at a point $a \in I$ iff for every seq^n $[a_n]$ in I converges to $a \Rightarrow \lim_{n\to\infty} f(a_n) = f(a)$

→ CONTINUOUS CRITERIA IN TERMS OF SEQUENCE

Proof: Let f be continuous at $a \in I$ & $[a_n]$ be a seq^n in I such that $a_n \longrightarrow a$.

Since f is continuous at a, therefore for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that: $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$ — ①

Again, $\lim_{n\to\infty} a_n = a \Rightarrow \exists$ a +ve integer m

such that $|a_n - a| < \delta,\ \forall\ n \geq m$ — ②

From ①, putting $x = a_n$, we have:

$|f(a_n) - f(a)| < \varepsilon$, when $|a_n - a| < \delta,\ \forall\ n \geq m$

$\Rightarrow |f(a_n) - f(a)| < \varepsilon,\ \forall\ n \geq m$ (by ②)

$\Rightarrow f(a_n) \longrightarrow f(a)$

Conversely,

Assume that f is not continuous at $x = a$, then we show that $\exists$ a seq^n $[a_n]$ in I converges to a, but $f(a_n) \not\longrightarrow f(a)$

Since f is not continuous at $x = a$, $\exists$ an $\varepsilon > 0$ such that for every $\delta > 0$, $\exists$ an $x \in I$ such that

$|f(x) - f(a)| \geq \varepsilon$, when $|x-a| < \delta$ — ③

by taking $\delta = 1/n$ we find for each +ve integer n, there is an $a_n \in I$ such that

$|f(a_n) - f(a)| \geq \varepsilon$ whenever $|a_n - a| < \frac{1}{n}$

$\Rightarrow f(a_n) \not\rightarrow f(a)$, as $n \longrightarrow \infty$

BMC 101

Discontinuity

f is discontinuous at a point $x=c$ if f is not continuous at $x=c$.

① Removable discontinuity
② Discontinuity of 1st kind
③ Discontinuity of 2nd kind

→ LHL = RHL ≠ f(a)

Ex - $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

→ $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

LHL ≠ RHL

→ one does not exist, or both do not exist

$f(x) = \begin{cases} 1, & x \in \mathbb{Q}^c \\ -1, & x \in \mathbb{Q} \end{cases}$

① Removable: A function f is said to have removable discontinuity at $x=c$ if $\lim_{x \to c} f(x)$ exist, but is not equal to the value f(a). Such a discontinuity can be removed by assigning a suitable value to the function at $x=c$

Ex- $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

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(1) **Ist kind:** f is said to have a discontinuity of Ist kind at $x=c$ if:

$\lim_{x \to c^-} f(x)$ & $\lim_{x \to c^+} f(x)$ both exist but are not equal

Ex - $f(x) = \begin{cases} 3, & x \geq 0 \\ -3, & x < 0 \end{cases}$

(2) **IInd kind:** f is said to have a discontinuity of 2nd kind at $x=c$ if: neither LHL nor RHL exist or either LHL or RHL exist

$\lim_{x \to c^-} f(x)$ nor $\lim_{x \to c^+} f(x)$ does not exist

(1) If LHL does not exist: Discontinuity of 2nd kind from left.

(2) If RHL does not exist: Discontinuity of 2nd kind from right.

⇒ **Questions:**

(1) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 3, & x = 1 \\ 4x, & 1 < x \leq 2 \end{cases}$, $x = 1$

First kind

(2) $f(x) = \begin{cases} +1, & x = Q^c \\ -1, & x = Q \end{cases}$

discontinuous on $\mathbb{R}$

2nd kind

write statement

Proof: Let $a \in Q$

$a_n \longrightarrow a \Rightarrow f(a_n) \longrightarrow f(a)$

irrationals

$f(a_n) = 1, f(a) = -1$

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Imp

Proof: Let a be any rational number:

$$f(a) = -1$$

For each +ve integer n, we can choose ~~one~~ irrational number a_n such that

$$|a_n - a| < \frac{1}{n}$$

$$\Rightarrow a_n \to a$$

But, $f(a_n) = 1, \forall n$ & $f(a) = -1, \forall n$

So that $\lim_{n \to \infty} f(a_n) \neq f(a)$

Hence, the given function is discontinuous at any $\mathbb{Q}$.

Next, let b be any irrational no.:

$$f(b) = 1$$

For each +ve integer n, we can choose ~~an~~ a rational number b_n:

$$|b_n - b| < \frac{1}{n}$$

$$\Rightarrow b_n \longrightarrow b$$

But, $f(b_n) = -1, \forall n$ & $f(b) = 1, \forall n$

So that ~~limit of~~ $\lim_{n \to \infty} f(b_n) \neq f(b)$

$\therefore$ The given function is discontinuous at any $\mathbb{Q}^c$

$\Rightarrow f(x)$ is discontinuous on $\mathbb{R}$

Imp

Ques 3) $f(x) = \begin{cases} x, & x = \mathbb{Q}^c \\ -x, & x = \mathbb{Q} \end{cases}$

Prove f is discontinuous at $x \neq 0$

f is continuous at $x = 0$

Imp

4) $f(x) = \begin{cases} x^2, & x \in \mathbb{Q}^c \\ -3x - 2, & x \in \mathbb{Q} \end{cases}$

$$x^2 = -3x - 2$$

$$x^2 + 3x + 2 = 0$$

$$x = -1, -2$$

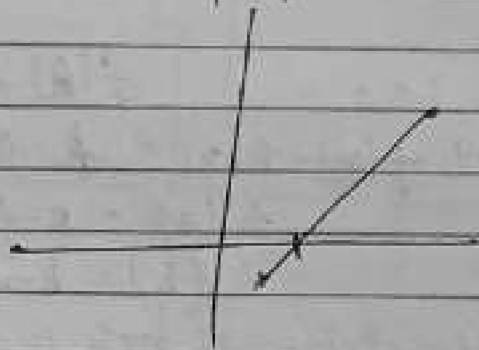
Location of Roots Theorem:

Let $I = [a,b]$, let $f: I \to R$ be a continuous function on I.

If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$ ~~then~~ there exists a point $c \in (a,b)$ such that $f(c) = 0$

Ex ① $f(x) = \sin x$

$I = [-\frac{\pi}{2}, \frac{\pi}{2}]$



$f(-\pi/2) = -1$

$f(\pi/2) = 1$

$f(0) = 0$

$\boxed{c = 0}$

$\boxed{c = n\pi}$

② $f(x) = \cos x$, $x \in [0, \pi]$

$f(0) = 1$

$f(\pi) = -1$

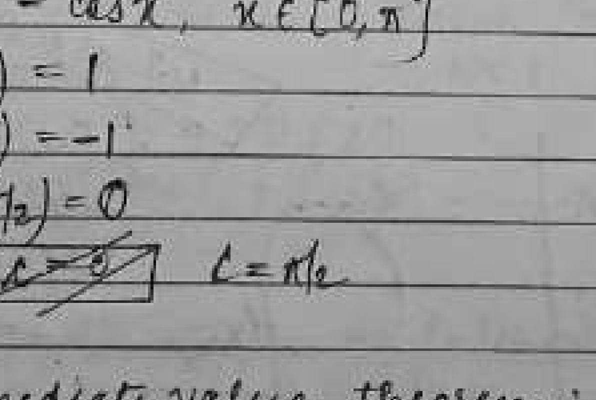
$f(\pi/2) = 0$

$\rightarrow \boxed{\cancel{c = 0}}$ $c = \pi/2$

Intermediate value theorem:

Let I be an interval & let $f: I \to R$ be a continuous function on I. If $a, b \in I$ & if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then there exists a point $c \in I$ ~~such that~~ between a & b such that $f(c) = k$

$f(b) < k < f(a)$



$f(a) < k < f(b)$

Proof ① Let $a<b$

Let $g(x) = f(x) - k$, then

$g(a) = f(a) - k < 0$

$g(b) = f(b) - k > 0$

Theorem $\exists\, c \to (a,b) : g(c) = 0$

$\Rightarrow f(c) = k$

② Let $b<a$ $\quad f(a) < k < f(b)$

Let ~~g(x)~~ $h(x) = k - f(x)$

$h(a) = k - f(a) \geq 0$

$h(b) = k - f(b) \leq 0$

By location of roots theorem, there exists a point $c \in (a,b)$: $f(c) = k$

$\Rightarrow h(c) = 0$

$\Rightarrow h(c) = k - f(c) = 0$

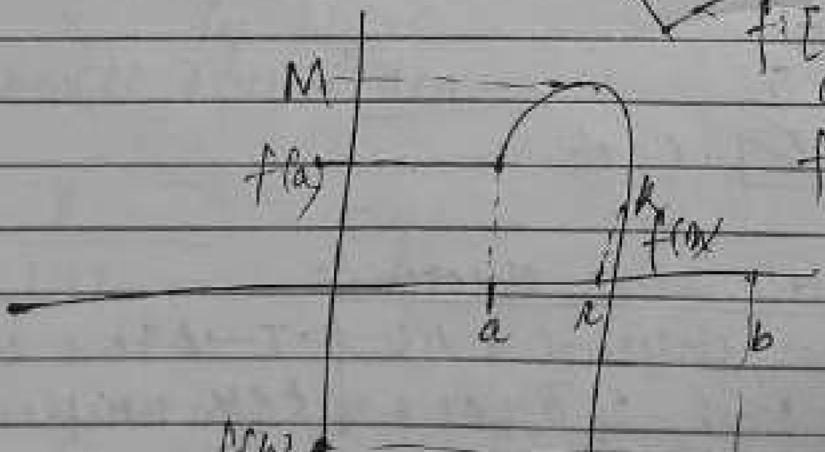
$= f(c) - k$

M.I. or M.D.

$f : [a,b] \to \mathbb{R}$ is continuous

$f([a,b])$ = Range

= Closed & bounded



$$\boxed{f([a,b]) = [m, M]}$$

In general.

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(3) $f(x) = x^2$

$I = [1, 2)$

$f([1,2)) = [1,4)$

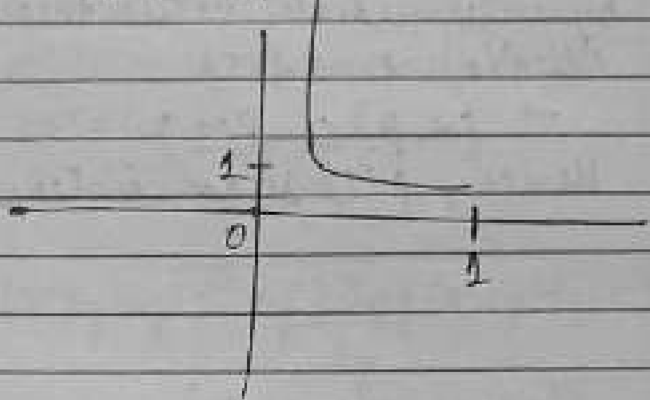
Range follows ~~clo~~ closure property / nature similar to domain

$\boxed{\text{Counter example}}$

(4) $f(x) = \frac{1}{x}$

$x \in (0, 1]$

$f(x) = [1, \infty)$



Preservation of Interval Theorem:

Let I be an interval & let $f: I \to \mathbb{R}$ be a continuous function on I.

Then $f(I)$ is an interval ($\Rightarrow$ not $\{\}$)

Eg (1) $f(x) = x^2$ — NOT INTERVAL

$I \in [0,1] \cup [2]$

$f(I) = [0,1] \cup \{4\}$

Note:

~~Proof~~ Characterization of Interval:

If S is a subset of $\mathbb{R}$ that contains atleast 2 points & has the following property:

If $x, y \in S$ & $x < y$

$\Rightarrow [x, y] \subseteq S$

Then S is an interval.

Proof: $f: I \to R$

$\boxed{f(I) \text{ is interval}}$ To prove

Let $\alpha, \beta \in f(I)$.

$\boxed{[\alpha, \beta] \subseteq f(I)}$ if $\alpha < \beta$

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Then show that $[\alpha, \beta] \subseteq f(I)$
If $\alpha, \beta \in f(I)$ then $\exists$ $a, b \in I$ such that $f(a) = \alpha$,
$f(b) = \beta$
Now, if $k \in (\alpha, \beta)$
$\alpha < k < \beta$
By intermediate value theorem, there exists a point
$c \in I$ $f(c) = k \in f(I)$
$\Rightarrow [\alpha, \beta] \subseteq f(I)$
Hence, $f(I)$ is an interval

BMC-101

* Uniform continuity: ⇒ $\delta = \delta(\varepsilon)$

↳ to check continuity on a set.

ε is same

δ changes

$f: A \to \mathbb{R}$

$\delta = \cancel{\delta(\varepsilon)} = \delta(\varepsilon, a)$ generally

Delta depends upon ε & a

$\delta = \delta(\varepsilon, a)$

$= \{\delta(\varepsilon, a) \mid a \in A\} = \inf\{\delta(\varepsilon, a) \mid a \in A\}$

δ uniform

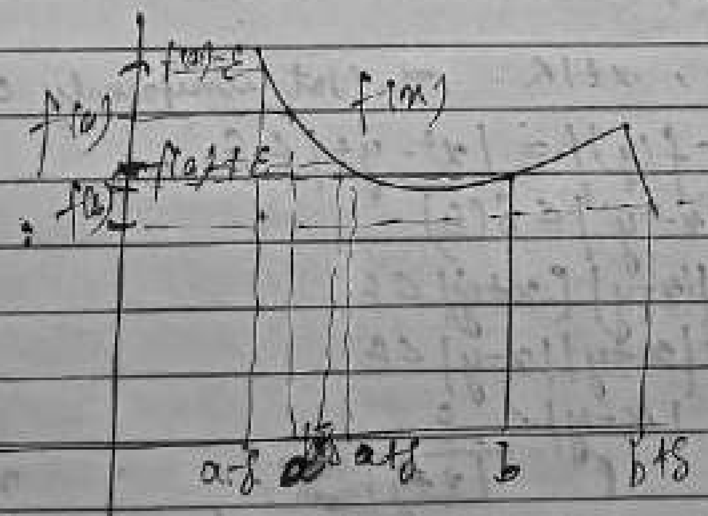
Continuity:

f is continuous at $x = a$ (ε-δ)

$\forall\ \varepsilon > 0,\ \exists$ a $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

$\varepsilon = 1/2$



$(f(a) - \varepsilon, f(a) + \varepsilon)$ $(a - \delta, a + \delta)$

$(f(b) - \varepsilon, f(b) + \varepsilon)$ $(b - \delta, b + \delta)$

$f: A \to \mathbb{R}$

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Uniform δ :

$\delta = \delta(\varepsilon)$

δ independent of 'a', only depends on ε

* Uniform Continuity:

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$. Then f is called uniformly continuous on A if for each $\varepsilon > 0$ $\exists$ a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any number satisfying $|x-u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

e.g. ① $f(x) = x$, $x \in \mathbb{R}$

$|f(x) - f(y)| = |x-y| < \varepsilon$

$\delta = \varepsilon$

$|x-y| < \delta$

② $f(x) = x^2$, $x \in \mathbb{R}$ $\to$ Not uniformly continuous

$|f(x) - f(y)| = |x^2 - y^2| < \varepsilon$

$|x-y| < \delta(\varepsilon)$

$|(x-y)(x+y)| < \varepsilon$

$= |x+y||x-y| < \varepsilon$

$|x-y| < \dfrac{\varepsilon}{|x+y|}$

$|x-y|\,|x+y| = |x-y|(|x|+|y|) < \varepsilon$

$|x-y| < \dfrac{\varepsilon}{|x|+|y|}$, $x, y \in \mathbb{R}$

$\delta_{uniform} = \inf\left\{\dfrac{\varepsilon}{|x|+|y|} \;\middle|\; x, y \in \mathbb{R}\right\}$

$\delta_{uniform} = 0$

$|\cdot| : \mathbb{R} \to [0, \infty)$

$\delta_{uniform} = \inf\left\{\dfrac{\varepsilon}{\infty} \to 0\right\}$

for bounded set $x \in A$

<u>bounded set</u>: $|x| \le M$, $M > 0$

$$\text{Suniform} = \left\{ \frac{\varepsilon}{|x|+|y|} \,\middle|\, x, y \in A \right\} = \frac{\varepsilon}{2M} > 0$$

③ $f(x) = x^2$ is U.C. on $\mathbb{R}$:

① on $[1, 1000]$

② on $(0, 10000)$

④ $f(x) = x^3 \cancel{+ x^2}$, $x \in \mathbb{R}$

⑤ $f(x) = x^4$, $x \in \mathbb{R}$

⑥ $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

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BMC 101

* Non Uniform Continuity Criteria :

Let $A \subseteq \mathbb{R}$ & let $f: A \to \mathbb{R}$. Then f is not U.C on A if $\exists$ $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points $x, u \in A$ s.t

$$|x-u| < \delta \text{ \& } |f(x) - f(u)| \geq \varepsilon_0$$

OR

There exists $\varepsilon_0 > 0$ & two sequences (x_n) & (u_n) in A such that $\lim (x_n - u_n) = 0$ & $|f(x_n) - f(u_n)| \geq \varepsilon_0$, $\forall$ $n \in \mathbb{N}$

eg. ① $f(x) = \frac{1}{x}$, $x \in (0,1)$ is not U.C.

$f(x)$ is continuous.

$x_n = \frac{1}{n}$ $\quad$ $u_n = \frac{1}{n+1}$

$$\boxed{\left|\frac{df}{dx}\right| \leq k \Rightarrow U.C}$$

Check: $\lim (x_n - u_n) = 0$

$$\frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \to 0$$

$f(x) = \frac{1}{x}$

$f(x_n) = f\left(\frac{1}{n}\right) = n$

$|f(x_n) - f(u_n)| = |n - n - 1| = 1$

$\varepsilon_0 = 1$

$|f(x_n) - f(u_n)| \geq \varepsilon_0$

② $f(x) = \sin x$, $x \in \mathbb{R}$

$f(x)$ is continuous

func^n must be bounded in its given domain

A func^n on $\mathbb{R}$ is unbounded

→ **Lipschitz function** ⇒ U.C.

Let $A \subseteq \mathbb{R}$ & $f: A \to \mathbb{R}$. Then f is called Lipschitz func^n if ∃ a constant $k > 0$: $|f(x) - f(y)| \le k|x-y|$, for all $x, y \in A$.
k = Lipschitz constant.

$$|f(x) - f(y)| \le k|x-y| < \varepsilon$$

$$|x-y| < \frac{\varepsilon}{k} = \delta$$

⇒ Every Lipschitz func^n is uniformly continuous.

$\boxed{\text{NOTE}}$ If f is differentiable then $\left|\frac{df}{dx}\right| \le k$ — ①

If f satisfies eq^n ①, then f is said to be Lipschitz function.

① $f(x) = \sin x, \cos x, e^x, x \in \mathbb{R}$ ⇒ continuous func^n

⇒ **Uniform Continuity Theorem**: Let I be a closed bounded interval & let $f: I \to \mathbb{R}$ be a continuous function on I. Then f is U.C. on I.

Ex: $I = (0,1)$, $f(x) = \frac{1}{x}$ ⇒ not U.C.

$I = [0, 100]$, $f(x) = x^2, e^x, \sin x, \cos x$ ⇒ U.C.

Proof – If f is not U.C. on I, then ∃ an $\varepsilon_0 > 0$ & seq^n $(x_n), (u_n) \in I$ such that ~~lim $(x_n - u_n) = 0$~~ $|x_n - u_n| < \frac{1}{n}$ &

$$|f(x_n) - f(u_n)| \ge \varepsilon_0, \forall n \in \mathbb{N}$$

Since I is bounded hence, sequence (x_n) is bounded then there is a subsequence (x_{n_k}) of (x_n) that converges to z. Since I is closed ⇒ $z \in I$.

$\Rightarrow$ If I is closed then the limit point of I belong within I

Now, $|u_{n_k} - z| \le |u_{n_k} - x_{n_k}| + |x_{n_k} - y| \to 0$

as $n \to \infty$

$\Rightarrow u_{n_k} \to y$, as $n \to \infty$

$\Rightarrow x_{n_k} \to y$ & $u_{n_k} \to y$

f is continuous then $\Rightarrow f(x_{n_k}) \to f(y)$ & $f(u_{n_k}) \to f(y)$

$\Rightarrow |f(x_{n_k}) - f(u_{n_k})| \to 0$, as $n \to \infty$

$\Rightarrow$ this is a contradiction of:

$$|f(x_{n_k}) - f(u_{n_k})| \ge \varepsilon_0, \forall n \in \mathbb{N}$$

★ Theorem:

If $f: A \to \mathbb{R}$ be a uniformly continuous func^n on A & if (x_n) is a cauchy sequence in A, then $f(x_n)$ is a cauchy sequence in $\mathbb{R}$.

Proof - Since f is U.C then $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$:

$|x - y| < \delta$ & $|f(x) - f(y)| < \varepsilon$, $\forall x, y \in A$

Also, since (x_n) is a cauchy seq. in A, then $\exists$ a +ve number $N(\delta) > 0$: $|x_n - x_m| < \delta$, $\forall n, m \ge N(\delta)$

by definition of U.C function:

$$|f(x_n) - f(x_m)| < \varepsilon, \forall n, m \ge N(\delta)$$

$\Rightarrow f(x_n)$ is a cauchy seq^n.

Ques ① $f(x) = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$ | ③ $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$

② $f(x) = \dfrac{1}{x^2}$, $x \in A = [1, \infty)$ & $B = (0, \infty)$

③ $f(x) = \sin(x^2)$, $x \in \mathbb{R}$

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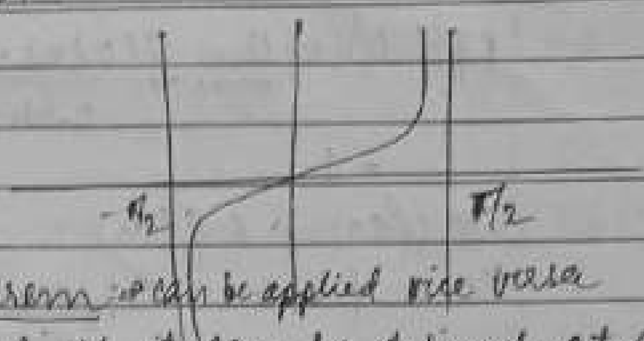
① $f(x) = \tan x$
$x \in (-\pi/2, \pi/2) \Rightarrow$ U.C.



*→ Continuous Extension Theorem → can be applied vice versa
A function is U.C. on (a,b) iff it can be defined at end points a & b such that the extended function is continuous on $[a,b]$.

$$\lim_{x \to \pi/2} \tan x = \infty$$

$f(x) = \frac{1}{x}$, $x \in (0,1)$ not continuous on $[0,1]$

*→ Differentiability:
Let $f: A \to \mathbb{R}$ be a real valued function & $A \subseteq \mathbb{R}$
Then the derivative of f at a point c is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$\text{LHD}: f'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{(c-h) - c}$$

$$\text{RHD}: f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{(c+h) - c}$$

f is differentiable at c if LHD = RHD

Ex ① ⇒ $f(x) = |x|$, $x \in \mathbb{R}$ is differentiable at $x = 0$?
→ $Lf'(x) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0-h-0} = \frac{f(-h) - f(0)}{-h} = \frac{|-h| - |0|}{-h}$
$= -1$