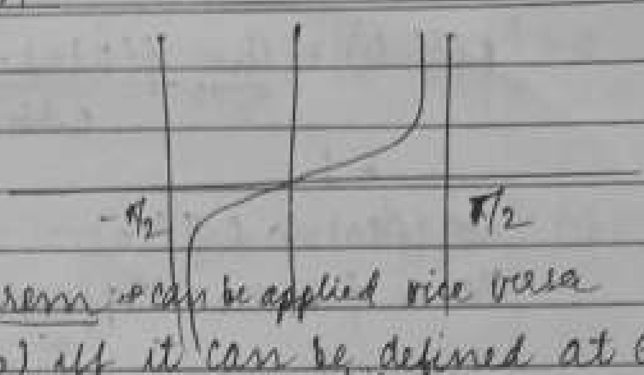
6+8 | ~~Aug~~ Sep — BMC 101

① $f(x) = \tan x$

$x \in (-\pi/2, \pi/2) \Rightarrow$ U.C.



*→ Continuous Extension Theorem → can be applied vice versa

A function is U.C. on (a,b) iff it can be defined at end points a & b such that the extended function is continuous on $[a,b]$.

$$\lim_{x \to \pi/2} \tan x = \infty$$

$f(x) = \frac{1}{x}$, $x \in (0,1)$ not continuous on $[0,1]$

*→ Differentiability:

Let $f: A \to \mathbb{R}$ be a real valued function & $A \subseteq \mathbb{R}$

Then the derivative of f at a point c is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$\text{LHD}: f'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{(c-h) - c}$$

$$\text{RHD}: f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{(c+h) - c}$$

f is differentiable at c if LHD = RHD

Ex ② ⇒ $f(x) = |x|$, $x \in \mathbb{R}$ is differentiable at $x = 0$?

$$\to Lf'(x) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0-h-0} = \frac{f(-h) - f(0)}{-h} = \frac{|-h| - |0|}{-h}$$

$$= -1$$

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$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0} = \frac{f(h)-f(0)}{h} = \frac{|h|-|0|}{h}$

$= 1$

$\Rightarrow Lf'(0) \neq Rf'(0)$.

Hence f is not differentiable at $x=0$.

(ii) at $x=1$

$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \frac{|1-h| - |1|}{-h} = \frac{|1|-|h|-|1|}{h}$

$= \frac{0}{h} = 0$

$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \frac{|1+h|-|1|}{h} = \frac{1-1}{h} = 0$

LHD = RHD

Hence, f is differentiable at $x=1$

(2) Check $f(x) = x^{1/3}$ at $x=0$

Ex(4) $f(x) = x^{1/3}$ at $x=0$ is continuous at $x=c$, $x \in \mathbb{R}$

$\rightarrow$ ~~$Lf'(0)$ $\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{(0-h)^{1/3}-(0)^{1/3}}{-h}$~~

$f'(0) = \lim_{h\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{1/3}-0}{x-0} = x^{1/3-1}$

$= x^{-2/3} = \frac{1}{x^{2/3}} = \infty$

(5) $f(x) = |x| + |x-1|$

let $x-1 = y$

$x = |y+1|$

$y=0$

$x=1$

(6) $f(x) = |x|^3$ at $x=0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|^3}{x} = \frac{x|x|^2}{x} = |x|^2 = 0$$

✗ → Continuity is <u>necessary</u> condition for differentiability but <u>not sufficient</u> condition.

$f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$

<u>Theorem</u> – If f is differentiable at a point c, then it is continuous at c. But continuous function need not be differentiable.

<u>Proof</u>: Let $f: A \to \mathbb{R}$ be a differentiable function at point c. Then for all $x \in A$, & $x \neq c$, we have $[x, c \in I]$.

$$f(x) - f(c) = \frac{f(x)-f(c)}{x-c}(x-c)$$

Applying limit $x \to c$ on both sides, we get

$$\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right]$$

$$\lim (f \cdot g) = \lim f \cdot \lim g$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} (x-c)$$

$$= f'(c) \cdot 0$$

$$\lim_{x\to c} [f(x) - f(c)] = f'(c) \cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

→ f is continuous at point $x = c$.

Continuity is not sufficient condition for differentiability.

Prove also

(7) $f(x) = |x|$; at $x=0$, is continuous but not differentiable.

(8) $f(x) = x^{1/3}$ is continuous at $x=0$, but not differentiable at $x=0$.

(1) Q A function f is defined as

$$f(x) = x^p \cos\frac{1}{x}, \; x \neq 0 \;\&\; f(0) = 0$$

Find the value of p for which f is differentiable

$$f(x) = \begin{cases} x^p \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

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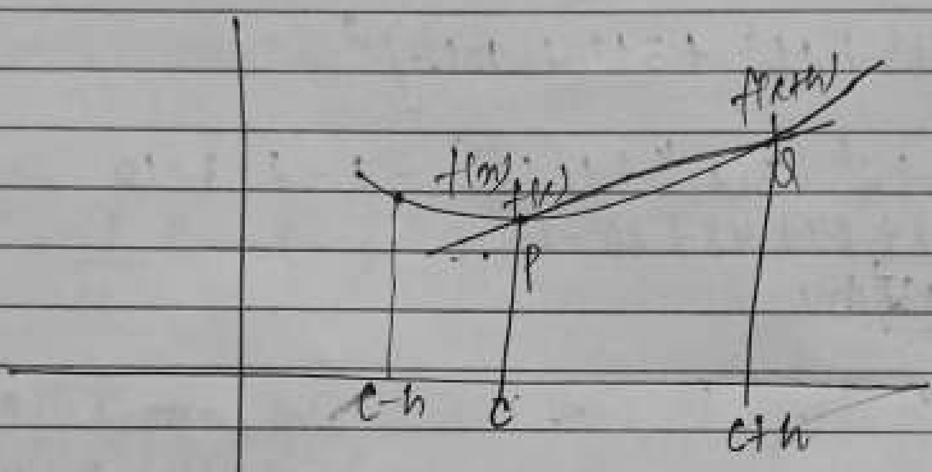
* Geometrical interpretation of derivative.

Slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\theta$

Slope represents variation of y with respect to x



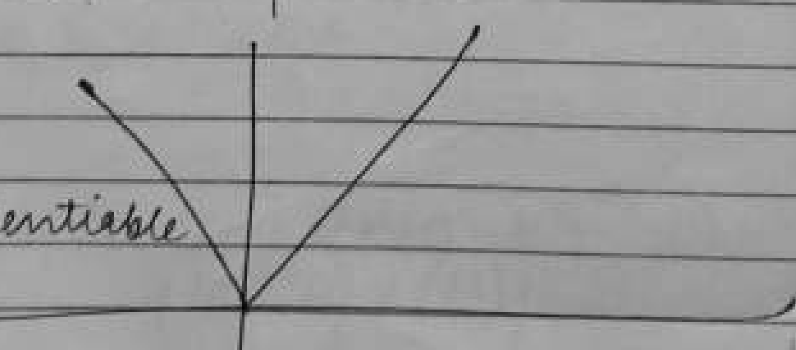
Slope $= \dfrac{f(c+h) - f(c)}{c+h-c} = \dfrac{f(c+h) - f(c)}{h}$

$f'(c) = \tan\psi$

① $f(x) = |x|$

$|x|$ at $x=0$

is non differentiable



$LHD = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{|x|}{x} = \dfrac{-x}{x}$

$= -1$

$RHD = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x}$

$= \dfrac{|x|}{x} = \dfrac{x}{x} = 1$

③ $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is diff. at $x=0$

$\Rightarrow$ differentiable ✓

④ $f(x) = |x^3| = |x^2 \cdot x| = |x^2| \cdot |x| = x^2|x|$

$$\lim_{x\to 0} \frac{x^2|x|-0}{x-0} = x|x| = 0$$

⑤ $f(x)=|x|^3 = |x|^2\cdot|x| = x^2\cdot|x|$

⑥ $f(x)=|x|+|x-1|+|x-100|+|x-2|^2$
not differentiable at 0, 1, 100

⑦ function not differentiable at 100 point.

⑧ $f(x)=|x|^{2n}, n\in\mathbb{N}$ = differentiable $= x^{2n}$

⑨ $f(x)=|x|^{2n-1}, n\in\mathbb{N}$ = differentiable but not a $n=1$

$f(x)=|x|$
$f(x)=|x|^2 = x^2$
$f(x)=|x|^3 = x^2\cdot|x|$
$f(x)=|x|^4 = x^4$
$f(x)=|x|^5 = x^4\cdot|x|$

⑩
$$f(x) = \begin{cases} x^p \cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Find:
(i) f is diff. at $x=0$
(ii) f is continuous at $x=0$

Sol^n- $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^p\cos\frac{1}{x} - 0}{x-0}$

$= \lim_{x\to 0} x^{p-1} \cos\frac{1}{x}$

$\boxed{p-1>0} \Rightarrow p>1 \Rightarrow$ differentiable

Continuity:

$\lim_{x \to 0} x^p \cos \frac{1}{x} = 0$

f is differentiable at $x=0$ if $p>1$

f is continuous at $x=0$ if $p>0$

$|f(x) - f(a)| < \varepsilon$ ~~for every~~ whenever $|x-a| < \delta$

$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0$

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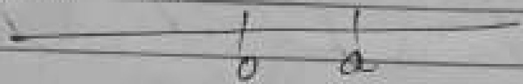
* **Relⁿ b/w Differentiability & Continuity:**

$y = f(x)$

$f: \mathbb{R} \to \mathbb{R}$



① **Continuous** $\Rightarrow$ $\boxed{\lim_{x \to a} f(x) = f(a)}$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

② **First Principle**: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f: \mathbb{R} \to \mathbb{R}$

$a \in \mathbb{R}$

diff.

then $\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)}$ TEST FOR DIFFERENTIABILITY

Ques $f: \mathbb{R} \to \mathbb{R}$

$f(x) = |x|$ at $x \to 0$

Solⁿ $\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = |0| = 0$

$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{|x| - |0|}{x - 0} = \frac{x}{x} = 1$

LHL: $\lim_{h \to 0} \frac{f(x-h) - f(x)}{x-h} = \lim_{h \to 0} \frac{|x-h| - |x|}{x-h}$

$= \lim_{h \to 0} \frac{|0-h| - |0|}{0-h} = \frac{-h}{-h} = +1$

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RHL: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{x+h} = \lim_{h\to 0} \frac{|x+h|-|x|}{x+h}$

$= \lim_{h\to 0} \frac{|0+h|-|0|}{0+h} = \lim_{h\to 0} \frac{|h|}{h} = \frac{h}{h} = 1$

Continuous

~~Not continuous~~ X [wrong]

[Remark] If a function is continuous at a point, it may or may not be differentiable at that point.

Continuity $\not\Rightarrow$ Differentiability

Theorem: A function is differentiable at a point $x=a$ then it will be continuous at $x=a$

Neccessary cond^n: Differentiability $\Rightarrow$ Continuity

If $p \Rightarrow q$
then not $q \Rightarrow$ not p

[Not continuous $\Rightarrow$ not differentiable]

Proof: Let a function $f(x)$ be differentiable at $x=c$.
Hence, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists

Now, $f(x)-f(c) = \frac{f(x)-f(c)}{(x-c)} \times (x-c)$, $(x \neq c)$

Taking limit as $x \to c$ we have ~~limit~~:
$\lim_{x\to c} [f(x) - f(c)]$

$= \lim_{x\to c} \frac{f(x)-f(c)}{(x-c)} \cdot \lim_{x\to c} (x-c)$

$= f'(c) \times 0 = 0$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0$

$\lim_{x \to c} f(x) - f(c) = 0$

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

★ **Differentiability & Monotonicity:**

Monotonically increasing means non-decreasing & vice versa.

★ **Monotonic Function:**

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be monotonic function if function f is either monotonic increasing or monotonic decreasing.

→ **Monotonic Increasing func^n:**

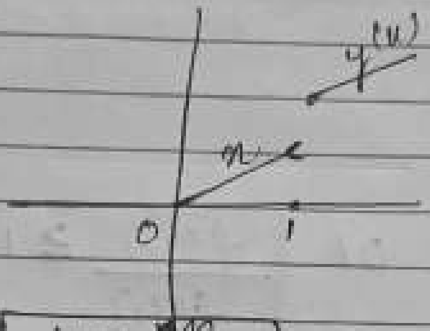
A function : $f: \mathbb{R} \to \mathbb{R}$ is said to be monotonically increasing func^n if $\forall$ x_1 & x_2 such that $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$

Strictly Increasing : $f(x_1) < f(x_2)$

→ **Monotonic Decreasing function:**

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be monotonically decreasing func^n if $\forall$ x_1 & x_2 such that $x_1 < x_2$ we have: $f(x_1) \geq f(x_2)$

Strictly Decreasing: $f(x_1) > f(x_2)$



$f(x) = x,\ x \in [0, 1)$
$f(x) = 1 - y^{(x)},\ x \in [1, \infty)$

$\boxed{\text{M.I. } f^n}$

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x$

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^3$

$f : [0, \infty) \to \mathbb{R}$
$f(x) = x^2$

$f(x) = e^x$

* All strictly increasing functions are monotonically increasing function.

* All strictly decreasing functions are monotonically decreasing function

$\boxed{\text{M.D. } f^n}$

$f : (0, \infty) \to \mathbb{R}$
$f(x) = \dfrac{1}{x}$

$\boxed{\text{M.I } f^n:}$
$f(x) = [x]$

$f(x) = e^{-x}$

$f(x) = -|x|$

* Can check by dividing or also by subtracting $\boxed{\text{M.I. } f^n}$
$f(x_1) \leq f(x_2)$ $\quad x_1 < x_2$

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Relation b/w Monotonicity & Differentiability :

① $f'(x)$ at $x = a \geq 0$
$\Rightarrow f(x)$ is monotonically increasing

② $f'(x)$ at $x = a \leq 0$
$\Rightarrow f(x)$ is monotonically decreasing.

③ $f'(x)$ at $x = a$ ~~≥0~~ > 0 $\quad f : \mathbb{R} \to \mathbb{R}$
$\Rightarrow f(x)$ is always $\quad f(x) = 2x$
strictly increasing

④ $f(x) = e^{x}$
$f : \mathbb{R} \to \mathbb{R}$
Strictly increasing func^n.

$\Rightarrow$ Differentiability: $f'(a) = \dfrac{f(a+h) - f(a)}{h}$

Using this we can prove that a function is monotonically increasing or decreasing

Ex - ① $\dfrac{f(a+h) - f(a)}{h} \leq 0$ monotonically ↓ing

② $\dfrac{f(a+h) - f(a)}{h} \geq 0$ monotonically ↑ing

③ Prove $f(x) = \log_e x$ is increasing or decreasing.

Sol^n Domain $\in (0, \infty)$
Codomain $= \mathbb{R}$
Range $= \mathbb{R}$
$\Rightarrow f : (0, \infty) \longrightarrow \mathbb{R}$

$f'(x) = \dfrac{1}{x}$

$\log a \rightarrow$ -ve value
$a < 1$

$$\frac{f(a+h)-f(a)}{h} = \frac{e^{(a+h)} - e^{a}}{h} = \frac{e^{a}\cdot e^{h} - e^{a}}{h}$$

$$= \frac{e^{a}(e^{h}-1)}{h}$$

strictly increasing

$$f'(x) = \frac{1}{x} > 0$$

(6) $f(x) = a^{x}$ $\boxed{0 < a < 1}$ $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f'(x) = \cancel{\frac{1 a^{x}}{x}}\, a^{x} \ln a$$

~~Non increasing, non decreasing~~

$\ln a \rightarrow$ always negative

Let $\ln a = -\alpha$

$$f'(x) = -\alpha a^{x}$$

$\Rightarrow f(x)$ is strictly decreasing.

(7) $f(x) = \sin x$

$f: \mathbb{R} \rightarrow \mathbb{R}$

$f'(x) = \cos x$

$-1 \leq \cos x \leq 1$

~~$[0, \pi/2] \rightarrow$ strictly increasing~~

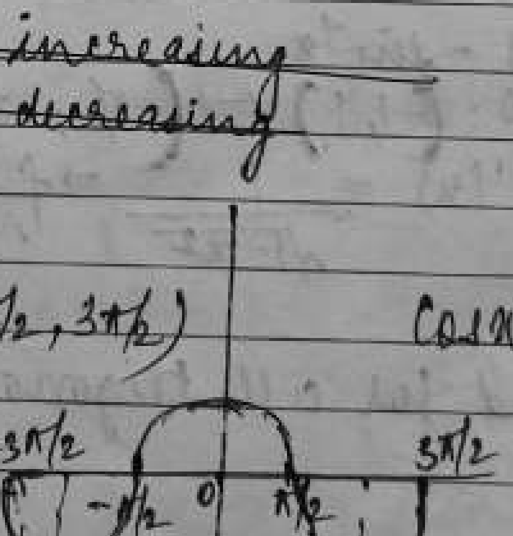
~~$[\pi/2, \pi] \rightarrow$ strictly decreasing~~

$-1 \leq \cos x < 0$

$x \in \cancel{(-\pi, \pi/2)}\ (\pi/2, 3\pi/2)$

$0 \leq \cos x \leq 1$

$x \in (-\pi/2, \pi/2)$



~~strictly monotonically~~

$(-\pi/2, \pi/2) \Rightarrow$ increasing strictly

$(\pi/2, 3\pi/2) \Rightarrow$ strictly decreasing

$f(x)$ will be monotonically increasing in $[-\pi/2, \pi/2]$

$\Rightarrow \uparrow$ in $\left[2\pi n - \pi/2, 2\pi n + \pi/2\right]$

$f(x)$ will be monotonically decreasing in $[\pi/2, 3\pi/2]$

$\Rightarrow \downarrow$ in $\left[2n\pi + \pi/2, 2n\pi + \frac{3\pi}{2}\right]$ $\boxed{\text{CHECK}}$

(8) $f(x) = \tan^{-1}x$

$f'(x) = \frac{1}{1+x^2} \geq 0$

strictly increasing

(9) $f(x) = \cos^{-1}x$

$f'(x) = -1/\sqrt{1-x^2} < 0$ $\Rightarrow f$ is strictly decreasing

$f: \cancel{(0,1)} \longrightarrow (-1,1) \longrightarrow$

(10) $f(x) = \sin^{-1}x$

$f: [-1,1] \longrightarrow \left(-\pi/2, \pi/2\right)$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow f$ is strictly increasing

(11) Find for all trigonometric functions.

$\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \sec x$

TRIGONOMETRIC IDENTITIES & FORMULAE

BMC101

→ Find dy/dx :-

① $y = \log\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$

$$\frac{dy}{dx} = \frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} \cdot \sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{1}{2}\cot\left(\frac{\pi}{4}+\frac{x}{2}\right) \cdot \sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$= \frac{1}{2\left(\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\left(\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)} = \frac{1}{\sin 2\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$= \operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \boxed{\sec x}$$

★ Successive Differentiation:

Some standard nth derivative of the function:

① When $y = e^{ax+b}$

$D^n(y)$ means nth differentiation of $y = \frac{d^n y}{dx^n}$

$D^n(y) = a^n e^{ax+b}$

Solⁿ: $D(y) = a e^{ax+b}$

$D^2(y) = a^2 e^{ax+b}$

$D^3(y) = a^3 e^{ax+b}$

$\vdots$

$D^n(y) = a^n e^{ax+b}$

★ can also be proved by Principle of Mathematical induction

Ex - ① $y = e^{2x}$, 100th differentiation.

$[D^n(y) = 2^n e^{2x}$

from $D^n(y) = a^n e^{ax+b}$

$\Rightarrow D^{100}(y) = 2^{100} e^{2x}$

② $y = (ax+b)^m$

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Sol^n: $D(y) = m(ax+b)^{m-1}(a)$

$D^2(y) = m(m-1)(ax+b)^{(m-2)}(a^2)$

$D^3(y) = m(m-1)(m-2)(ax+b)^{(m-3)}(a^3)$

$\vdots$

$D^n(y) = m(m-1)(m-2)\cdots(m-(n-1))(ax+b)^{(m-n)}(a^n)$

(m) is ~~less~~ greater than $(n) \Rightarrow m > n$

$\therefore D^n(y) = m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{(m-n)}(a^n)$

Particular case:

1) If m is a positive integer then:

$$= \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(m-n-1)\cdots 2\times 1}{(m-n)(m-n-1)\cdots 2\times 1}$$

$$= \boxed{\frac{m!}{(m-n)!}(ax+b)^{(m-n)}a^n}$$

2) If $m < n$ then $D^n(y) = y^{(n)} = 0$

3) When m is a positive integer & $m = n$

$\Rightarrow D^n(y) = a^n \frac{n!}{0!}(ax+b)^0 = a^n n! = a^m m!$

*** Most Imp

4) If m is a negative integer then: $m = -p$, $p \in \mathbb{Z}^+$

Then: $D^n((ax+b)^m) = D^n\left((ax+b)^{-p}\right)$

$= -p(-p-1)(-p-2)\cdots(-p-n+1)(ax+b)^{(-p-n)}a^{n}$

$= (-1)^n p(p+1)(p+2)\cdots(p+n-1)(ax+b)^{-(p+n)}a^n$

$$= \frac{(-1)^n p(p+1)(p+2)\cdots(p+n-1)\cancel{(p+n-2)}(p-1)(p-2)\cdots 2\times 1\,(ax+b)^{-(p+n)}a^n}{(p-1)!}$$

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$$= \boxed{(-1)^n \frac{(p+n-1)!}{(p-1)!} (ax+b)^{-(p+n)} a^n}$$

Que - Find the nth derivative of:

1) $\frac{1}{(x-a)} = y$

2) $\frac{1}{(x-1)^3} = y$

1) $\cancel{D^n(y) = 0}$ $D^n(y) = (-1)^n \frac{(1+n-1)!}{(1-1)!} (x-a)^{-(1+n)} \cancel{a} 1^n$

$$= (-1)^n \, n! \, (x-a)^{-(1+n)}$$

2) $y = (x-1)^{-3}$

$$D^n y = (-1)^n \frac{(3+n-1)!}{(3-1)!} (\cancel{ax+} x-1)^{-(3+n)} 1^n$$

$$D^n y = (-1)^n \frac{(n+2)!}{2!} (x-1)^{-(3+n)}.$$

~~Do this~~

3) $y = \frac{1}{(x-1)^3 (x-2)}$

Sol^n - $(x-1)^3 = x^3 - 1 - 3x(x-1)$

$$= (x^3 - 1 - 3x^2 + 3x)(x-2)$$

$$= x^4 - x - 3x^3 + 3x^2 - 2x^3 + 2 + 6x^2 - 6x$$

$$= x^4 - 5x^3 + 9x^2 - 7x$$

4) $\frac{1}{(x+1)(x-2)} = y$

sol^n: $\cancel{(x-1)}\ \cancel{D}y = \frac{1}{\cancel{x^2-3x+2}}$

$$\frac{1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$1 = A(x-2) + B(x-1)$$

$\cancel{1 = Ax}$

① $x = 2$

$$\boxed{1 = B}$$

② $x = 1$

$$1 = -A$$

$$\boxed{A = -1}$$

$$\underset{a}{\frac{-1}{(x-1)}} + \underset{b}{\frac{\cancel{B}\,1}{(x-2)}}$$

$$\boxed{D^n(a) = (-1)^{n+1}\, n!\,(x-1)^{-(1+n)}}$$

$$\boxed{D^n(b) = (-1)^{n\cancel{+1}}\, n!\,(x-2)^{-(1+n)}}$$

$$\Rightarrow (-1)^n\left[(-1)\,n!\,(x-1)^{-(1+n)} + n!\,(x-2)^{-(1+n)}\right]$$

$$\boxed{= (-1)^n(n!)\left[-(x-1)^{-(1+n)} + (x-2)^{-(1+n)}\right]}$$

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① $y = \log(ax+b)$

Sol^n: $D^1 y = \frac{1a}{ax+b}$

$D^2 y = a\frac{d(ax+b)^{-1}}{dx} = -a^2(ax+b)^{-2} = (-1)\,a^2(ax+b)^{-2}$

$D^3 y = 2a^3(ax+b)^{-3}$

$D^4 y = -6a^4(ax+b)^{-4}$

$D^5 y = 24\,a^5\,(ax+b)^{-5}$

$D^n y = (\cancel{-1+}n)!\,a^n(ax+b)^{-n}(-1)^{n-1}$

$$\boxed{D^n y = \frac{(n-1)!\,a^n\,(-1)^{n-1}}{(ax+b)^n}}$$

② Find the nth differentiation of $y = \sin(ax+b)$.

Sol^n: $D^1 y = a\cos(ax+b) = a\sin(ax+b+\pi/2)$

$D^2 y = (-1)^1 a^2 \sin(ax+b)$

$D^3 y = (-1)^1 a^3 \cos(ax+b)$

$D^4 y = (-1)^2 a^4 \sin(ax+b)$

~~①~~ when n is odd:

$D^n y = a^n \sin(ax+b)$

Sol^n: $D^1 y = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$D^2 y = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$

$D^3 y = a^3\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$

$$\boxed{D^n y = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)}$$

(3) $y = \cos(ax+b) = \sin(ax+b+\pi/2)$

Sol^n: $Dy = a\cos\left(ax+b+\frac{\pi}{2}\right) = a\sin\left(ax+b+\frac{2\pi}{2}\right)$

$D^2y = a^2\cos\left(ax+b+\frac{2\pi}{2}\right) = a^2\sin\left(ax+b+\frac{3\pi}{2}\right)$

$\vdots$

$D^ny = a^n\sin\left(ax+b+\frac{(n+1)\pi}{2}\right) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

Solve by other method.

* If $y = e^{ax}\cos(bx+c)$

$\Rightarrow y^{(n)} = r^n e^{ax}\cos(bx+c+n\phi)$

where, $r^2 = a^2+b^2$

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

* If $y = e^{ax}\sin(bx+c)$

$\Rightarrow y^n = r^n e^{ax}\sin(bx+c+n\phi)$

Where, $r^2 = a^2+b^2$

$\phi = \tan^{-1}(b/a)$

Ques (4) Find the nth differential coefficient of $\sin ax \cdot \sin bx$

Sol^n: $\sin(A+B) + \sin(A-B) = 2\sin A\,\cancel{\sin B}\cos B$.

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\Rightarrow \sin ax \cdot \sin bx = \cos(a-b)x - \cos(a+b)x$

$= \frac{(a-b)^n}{2}\cos\left(ax-bx+\frac{n\pi}{2}\right) - \frac{(a+b)^n}{2}\cos\left(ax+bx+\frac{n\pi}{2}\right)$

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(5) Find the nth differential coefficient of y:

$y = e^{ax} \sin bx \cos cx$

Sol^n: $\sin bx \cos cx = \{(\sin(bx+cx) + \sin(bx-cx))\} \div 2$

$D^n y = \frac{1}{2}(b+c)^n \sin\left(bx+cx+\frac{n\pi}{2}\right) + \frac{(b-c)^n}{2} \sin\left(bx-cx+\frac{n\pi}{2}\right)$

$\to y = e^{ax} \sin bx \cos cx$

$$D^n y = e^{ax}\frac{\sin(bx+cx)}{2} + \frac{e^{ax}}{2}\sin(bx-cx)$$

$$D^n y = \frac{1}{2}\left[e^{ax}\{\sin(b+c)x + \sin(b-c)x\}\right]$$

$$D^n y = \frac{1}{2}\left[e^{ax}\sin(b+c)x + e^{ax}\sin(b-c)x\right]$$

$$D^n y = \frac{1}{2}\left[r_1^n e^{ax}\sin(bx+cx+n\phi_1) + r_2^n e^{ax}\sin(bx-cx+n\phi_2)\right]$$

$r_1^2 = \cancel{a^2+b^2}\ (b+c)^2 + a^2$

$\phi_1 = \tan^{-1}\left(\frac{b+c}{\cancel{b}+a}\right)$

$r_2^2 = a^2 + (b-c)^2$

$\phi_2 = \tan^{-1}\left(\frac{b-c}{a}\right)$

★ <u>Lebanese Theorem:</u>

If ~~small~~ u & v are the func^n of x then:

$$D^n(u\cdot v) = (D^n u)\cdot v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)\cdot(D^2 v) + \dots + {}^nC_r(D^{n-r}u)(D^r v) + \dots + u(D^n v)$$

$\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$ $\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$

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⑥ If $y = x^2 e^x$, then prove that:

$$y^{(n)} = \frac{1}{2} n(n-1) \frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y$$

Solⁿ: $D^n(u \cdot v) = (D^n u)\cdot v + {}^nC_1 (D^{n-1}u)(D^1 v) + {}^nC_2 (D^{n-2}u)$
$(D^2 v) + \dots + {}^nC_r (D^{n-r}u)(D^r v) + \dots$
$+ u\cdot(D^n v)$

$y = x^2 e^x = e^x \cdot x^2$

$\Rightarrow D^n(x^2 e^x)$

$= (D^n x^2)\cdot e^x + {}^nC_1 (D^{n-1}x^2)(D^1 e^x) + {}^nC_2 (D^{n-2}x^2)$
$(D^2 e^x) + \dots + {}^nC_r (D^{n-r}x^2)(D^r e^x) + \dots + x^2\cdot(D^n e^x)$

$y^2(x^2\cdot e^x) = (D^2 x^2)\cdot e^x + {}^2C_1 (D^1 x^2)(D^1 e^x)$

$\Rightarrow D^n(x^2\cdot e^x) = D^n\{e^x \cdot x^2\}$

$\Rightarrow \frac{d^2y}{dx^2} = (D^2(e^x)\cdot x^2) + {}^2C_1 (D^1 e^x)(D^1 x^2) + {}^2C_2 (D^0 e^x)$
$(D^2 x^2)$

$\rightarrow \frac{d^2y}{dx^2} = x^2 \cdot e^x + \frac{2!}{1!}(e^x)(2x) + \frac{2!}{0!\,2!}(e^x)(2)$

$= x^2\cdot e^x + 4e^x\cdot x + 2e^x$

$\Rightarrow \frac{dy}{dx} = (De^x)\cdot x^2 + {}^1C_1 (D^0 e^x)(D^1 x^2)$

$\rightarrow \frac{dy}{dx} = x^2\cdot e^x + 2x\cdot e^x$

$\Rightarrow D^1 x^2 = 2x$	$D^n e^x = e^x$
$D^2 x^2 = 2$	
$D^3 x^2 = 0$	

$\Rightarrow D^n(e^x\cdot x^2) = e^x\cdot x^2 + n e^x\cdot 2x + n(n-1)e^x\cdot 2 +$
$n(n-1)(n-2)e^x \times 0$

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(7) Find the nth differential coefficient:

① $y = x^3 \cos x$

② $y = e^x \log x$

③ $y = x^3 \log x$

(8) If $y = x^{n-1} \log x$
Prove that $y^{(n)} = \dfrac{(n-1)!}{x}$

6) continued ...

$$D^n(e^x \cdot x^2) = x^2 e^x + 2nxe^x + 2n(n-1)e^x$$

$$y^{(n)} = \frac{1}{2} n(n-1)(x^2 \cdot e^x + 4e^x \cdot x + 2e^x) - n(n-2)$$
$$(x^2 \cdot e^x + 2xe^x) + \frac{1}{2}(n-1)(n-2)x^2e^x$$

$$y^{(n)} = \frac{1}{2} n(n-1)(x^2 \cdot e^x + 4e^x \cdot x + 2e^x) - n(n-2)(x^2 \cdot e^x +$$
$$2x \cdot e^x) + \frac{1}{2}(n^2 - n - 2n + 2)x^2 e^x$$

$$y^{(n)} = \frac{(n^2 - n)}{2}(x^2 e^x + 4e^x \cdot x + 2e^x) - (n^2 - 2n)(x^2 e^x + 2x \cdot e^x)$$
$$+ \frac{n^2(x^2 e^x)}{2} - \frac{3n(x^2 e^x)}{2} + x^2 e^x$$

$$y^{(n)} = \frac{n^2(x^2 e^x)}{2} - \frac{n(x^2 e^x)}{2} + 2n^2(e^x x) + n^2(e^x) - 2n(e^x \cdot x)$$
$$- n(e^x) - n^2(x^2 \cdot e^x) - 2n^2(x \cdot e^x) + 2n(x^2 e^x)$$
$$+ 4n(x \cdot e^x) + n^2 \frac{(x^2 e^x)}{2} - \frac{3n(x^2 e^x)}{2} + x^2 e^x$$