Integral transforms are widely used in the solution of Partial differential Equations

The choice of a particular transform for the solution depends upon the nature of the boundary conditions of the equations and the facility with which the transform $f(s)$ can be inverted to giv the original function $F(t)$

The Integral transform $f(s)$ of a function $F(t)$ is defined as

$$I[F(t)] = f(s) = \int_a^b F(t)\, K(s,t)\, dt$$

when $K(s,t) = e^{-st}$, the integral transform is defined by Laplace transform

$$L[F(t)] = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

2. when $K(s,t) = e^{ist}$, the
transform is defined as Fourier
transform ie

$$F[F(t)] = f(s) = \int_{-\infty}^{\infty} F(t)\, e^{ist}\, dt$$

3. when $K(s,t) = \sin st$, the integral
transform is defined to be Fourier sine
transform ie

$$F_s[F(t)] = f(s) = \int_{0}^{\infty} F(t) \sin st\, dt$$

4. when $K(s,t) = \cos st$
the integral transform is defined as
Fourier Cosine transform

$$F_c[F(t)] = f(s) = \int_{0}^{\infty} F(t) \cos st\, dt$$

5. Hankel transform by

$$H[F(t)] = f(s) = \int_{0}^{\infty} t\, J_n(st)\, F(t)\, dt$$

6. Mellin transform by

$$M[F(t)] = f(s) = \int_{0}^{\infty} F(t)\, t^{s-1}\, dt$$

Transform or

If $F(x)$ be a function defined on $(-\infty,\infty)$ & be piecewise continuous in each partial interval & absolutely integrable in $(-\infty,\infty)$ then the Fourier transform of $F(x)$ is a function of new variable s and it is denoted by

$$F\{F(x)\} = \bar{F}(s) = f(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx \rightarrow (1)$$

Then function $F(x)$ is then called inverse Fourier transform of $\bar{F}(s)$ or $f(s)$ and is denoted by

$$F(x) = F^{-1}\{\bar{F}(s)\} \quad \text{or} \quad F(x) = F^{-1}\{f(s)\} \rightarrow (2)$$

Inversion formula for Fourier Transform or Complex Fourier Transform

If $\bar{F}(s)$ is the Fourier transform of $F(x)$ and if $F(x)$ satisfy the Dirichlet conditions in every finite interval $(-l, l)$ and further if $F(x)$ is absolutely integrable in $(-\infty,\infty)$, then at every point of continuity of $F(x)$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\, e^{-isx}$$

$F^{-1}\{f(s)\} =$

Proof → We know that complex Fourier integral formula is given by

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\underline{\int_{-\infty}^{\infty} F(u)\, e^{isu}\, du}\right\} ds$$

We know that complex Fourier transform of $F(x)$ is

$$\bar{F}(s) = \int_{-\infty}^{\infty} e^{isx} F(x)$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(s)\, e^{-isx}\, ds$$

which is the required inversion formula for complex Fourier transform

Thus we have

$$F\{F(x)\} = \bar{F}(s) = f(s) = \int_{-\infty}^{\infty} e^{isx}\, F(x)\, dx$$

$$\& \quad F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, \bar{F}(s)\, d\cancel{x}$$

$$\text{or } F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, f(s)\, ds$$

transform of $f(x)$

if $f(x) = \begin{cases} e^{iwx} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$

Solution → We Know that Fourier Complex Transform

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{a} e^{isx} f(x)\, dx + \int_{a}^{b} e^{isx} f(x)\, dx + \int_{b}^{\infty} e^{isx} f(x)\, dx$$

$$= 0 + \int_{a}^{b} e^{isx} e^{iwx}\, dx + 0$$

$$= \int_{a}^{b} e^{i(s+w)x}\, dx$$

$$= \left[\frac{e^{i(s+w)x}}{i(s+w)} \right]_{a}^{b}$$

$$= \frac{1}{i(s+w)} \left[e^{i(s+w)b} - e^{i(s+w)a} \right]$$

$$= \frac{i}{s+w} \left[e^{i(s+w)a} - e^{i(s+w)b} \right]$$

Ans.